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OVALS, DUALITIES, AND DESARGUES'S THEOREM

T. G. OSTROM

Introduction. Consider a projective plane with $n + 1$ points on each line, An *oval* $\mathfrak{C}$ is a set of $n + 1$ points, no three of which are collinear.

DEFINITION. (1) A line which contains two points of $\mathfrak{C}$ will be called a *secant* of $\mathfrak{C}$.

(2) A line which contains exactly one point of $\mathfrak{C}$ will be called a *tangent* of $\mathfrak{C}$ or an *absolute line.*

(3) A line which contains no points of $\mathfrak{C}$ will be called an *exterior line* of $\mathfrak{C}$.

Each point of $\mathfrak{C}$ lies on exactly one tangent of $\mathfrak{C}$. Qvist **(10)** has shown that, for n odd, no three tangents of $\mathfrak{C}$ are concurrent. For n odd, we can make the following definitions.

DEFINITION. (4) A point which lies on two tangents of $\mathfrak{C}$ will be called an *exterior point* of $\mathfrak{C}$.

(5) A point which lies on no tangents of $\mathfrak{C}$ will be be called an *interior point* of $\mathfrak{C}$.

(6) Points of $\mathfrak{C}$ will sometimes be referred to as *absolute points.*

There is a natural correspondence between secants and exterior points. Each secant contains two absolute points; by taking the point of intersection of their tangents a definite exterior point is determined corresponding to the secant. Let the point corresponding to a given secant be called the *pole* of the secant, the secant will be called the *polar* of the point. The number of exterior points and the number of secants are both equal to $\frac{1}{2}n(n + 1)$ (the number of pairs of absolute points). Since there are $n^2 + n + 1$ points in all, $n + 1$ of which are absolute, the number of interior points is $\frac{1}{2}n(n - 1)$. Now Baer **(1)** has shown that the absolute points of a polarity form an oval unless n is a square or is even. Segre has shown that, in a Desarguesian plane, every oval determines a conic. It is well known that a conic in a Desarguesian plane determines a polarity.

The following form of Desargues's theorem holds in any plane admitting a polarity: "If two self-polar triangles are perspective from a point, they are perspective from the polar of that point." While we do not restrict ourselves to Desarguesian planes, we find that Desargues's configuration comes up again and again for suitably restricted pairs of triangles.

In §1, we study certain kinds of collineations which carry an oval into itself. In §2, we study certain respects in which an oval resembles a conic in a Desarguesian plane, especially in its relations to a modified type of harmonic set. In § 3, we specialize the results of §2 to the case where the oval consists of the absolute points of a polarity. In §4, we consider ovals in cyclic and other

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transitive planes. In §5, we show the existence of Fano's configuration for certain planes with n even. Except in §5, we restrict ourselves to odd values of n. The author is greatly indebted to the referee for suggestions which led to the inclusion of §2.

1. Collineations carrying an oval into itself. In this part, we consider collineations σ satisfying the following conditions: (1) σ must carry some oval $\mathfrak{C}$ into itself. (2) Excepting the identity, no power of σ leaves fixed any points which are not left fixed by σ itself.

σ, then, may leave certain points fixed, but the other points are arranged in cycles whose length is equal to the order of σ.

We list some fairly obvious properties of σ:

1.1. Any collineation which carries $\mathfrak{C}$ into itself must carry exterior points of $\mathfrak{C}$ into exterior points and interior points into interior points.

1.2. Since the g.c.d. of $n+1$, $\frac{1}{2}n(n+1)$ and $\frac{1}{2}n(n-1)$ is 1, σ must leave at least one point fixed.

1.3. If an exterior point U is fixed, its polar is fixed, and vice versa. This follows from the fact that the two tangents through U must either be fixed or mapped into each other. In turn, the corresponding points of tangency must either be fixed or mapped into each other.

1.4. If the fixed points of σ constitute a proper subplane with n_1+1 points on a line, and one of the fixed points belongs to $\mathfrak{C}$, then n_1+1 points of $\mathfrak{C}$ are fixed and the fixed points of $\mathfrak{C}$ constitute an oval of the subplane.

Proof. Let A be a fixed point of $\mathfrak{C}$. There are n_1+1 fixed lines through A, one of which will be the tangent at A. The other will be secants. But if we have a fixed secant through a fixed absolute point, the other absolute point on the secant must also be fixed.

1.5. If three points of $\mathfrak{C}$ are fixed, the fixed points of the plane constitute a proper subplane, since the poles of the corresponding secants must also be fixed.

1.6. If (a) no points (b) one point (c) two points or (d) n_1+1 points of $\mathfrak{C}$ are fixed, then the order of σ divides (a) $n+1$, (b) n, (c) $n-1$, or (d) $n-n_1$. This is essentially a generalization of some of the results in **(9)**. See **(2,** Theorem 3**)** for a further characterization of σ if the order of σ is a prime power.

It should perhaps be noted that properties 1.1, 1.3, 1.4, and 1.5 apply to any collineation carrying $\mathfrak{C}$ into itself.

THEOREM 1.1. *If σ leaves invariant a proper subplane with n_1+1 points on a line, and if all of the fixed points are interior, then the order of σ divides*

$$(\tfrac{1}{2}(n+1),\ n_1+1).$$

Proof. If a secant were fixed, the exterior point which is its pole would also be fixed. Hence there are no fixed secants. Similarly, no tangents are

fixed. Hence the fixed lines are all exterior lines. Since each exterior point lies on two of the $n+1$ tangents, it follows that half of the points on an exterior line are exterior and the other half are interior points. (Similarly, a secant contains $\frac{1}{2}(n-1)$ exterior points and $\frac{1}{2}(n-1)$ interior points). Consider a fixed exterior line u. The order of σ must divide the number of non-fixed interior points on u, which is $\frac{1}{2}(n+1)-(n_1+1)$. But the number of non-fixed exterior points on u is $\frac{1}{2}(n+1)$, so that the order of σ must also divide $\frac{1}{2}(n+1)$.

THEOREM 1.2. *If σ is of even order, and leaves invariant a proper subplane with n_1+1 points per line, at least one point of which belongs to $\mathfrak{C}$, then $n=n_1^2$ and none of the fixed points is interior.*

Proof. If σ is of even order, some power of σ is of order 2 and has the same fixed points. Without loss of generality, take σ to be of order 2. By property 4, n_1+1 points of $\mathfrak{C}$ are fixed. These n_1+1 points determine $\frac{1}{2}n_1(n_1+1)$ fixed secants. In addition, the remaining $n-n_1$ points of $\mathfrak{C}$ are interchanged in pairs, thus determining $\frac{1}{2}(n-n_1)$ fixed secants. Each fixed secant determines a fixed point. The number of fixed interior points is then

$$n_1^2+n_1+1-(n_1+1)-\tfrac{1}{2}n_1(n_1+1)-\tfrac{1}{2}(n-n_1)=\tfrac{1}{2}(n_1^2-n).$$

But **(6)** $n \geqslant n_1^2$.

This theorem would seem to be closely related to Mann's theorem **(7)** that a cyclic plane possesses a multiplier of even order only if n is a square, since, as we shall see, there are ovals in cyclic planes which are left invariant by the multipliers. We have been unable to obtain any interesting results for the case where no point of $\mathfrak{C}$ is fixed except that $\frac{1}{2}(n+1)$ exterior points remain fixed if the order of σ is even.

THEOREM 1.3. *If the fixed points of σ consist of the points on a line u and a point U not belonging to u or $\mathfrak{C}$, then* (a) *σ is of order 2;* (b) *every triangle and its image satisfy Desargues's theorem;* (c) *if U is exterior, u is its polar;* (d) *if U is exterior and AB, CD are two secants through U, where A, B, C, and D are absolute points, then the secants AC and BD intersect in a point of u, similarly, AD and BC intersect in a point of u.*

Proof. (a) Every line through U is fixed. Let AB be a fixed secant through U. Then $A \rightleftarrows B$, or A and B are fixed. But at most two points of $\mathfrak{C}$ lie on u and there are at least $\frac{1}{2}(n-1)$ (fixed) secants through U. Hence at least one pair of absolute points on a secant must be interchanged. (b) Let V, W, X be three non-fixed collinear points and let V_1, W_1, and X_1 be their images. Then the lines VV_1, WW_1, and XX_1 are fixed lines and must intersect in U. The lines VW and V_1W_1 are each other's images and must intersect in a point on u. Similarly, the other pairs of corresponding sides intersect in a point on u. (c) If U is exterior, its polar is a fixed line not through U and hence must be u. (d) Since $A \rightleftarrows B$ and $C \rightleftarrows D$, it follows that $AC \rightleftarrows BD$ and $AD \rightleftarrows BC$ so that these pairs of lines must intersect in u.

THEOREM 1.4. *Suppose that, for every point U not on an oval $\mathfrak{C}$, there is a collineation taking $\mathfrak{C}$ into itself which leaves U and every point on some line u fixed (u depending on U) then $\mathfrak{C}$ determines a polarity.*

Proof. Since all secants through U are fixed, their poles are fixed and lie on the line u. If U is interior, define the polar of U to be u, U the pole of u. With this extended relation of pole and polar, it is still true that a line is fixed if and only if its pole is fixed. Now, if we consider all lines through *any* point U, their poles must lie all on a certain line u. Note that only one such collineation is possible for each point U, since if there were two such collineations σ_1 and σ_2 then $\sigma_1 \sigma_2^{-1}$ would leave every point of $\mathfrak{C}$ fixed and $\sigma_1\sigma_2^{-1}$ is the identity.

2. Harmonic sets and harmonic ovals.

DEFINITION. An ordered set of collinear points $\{AB: CD\}$ will be said to be *harmonic with respect to the oval* $\mathfrak{C}$ if (1) A and B belong to $\mathfrak{C}$, (2) C is an exterior point, and (3) D is on the polar of C. When only one oval is under discussion, we shall merely speak of a harmonic set of points. The point D will sometimes be referred to as the fourth harmonic point or the harmonic conjugate of C.

DEFINITION. A perspectivity between the points of a secant and the points of a secant will be said to be a *perspectivity on* $\mathfrak{C}$ if either (1) the point of intersection of the secants is not absolute and absolute points correspond to absolute points, or (2) the secants intersect in an absolute point A, the center of perspectivity is on the tangent at A, and absolute points correspond to absolute points. The symbol $\overline{\wedge}$ between two sets of points will indicate that they are related by a perspectivity on $\mathfrak{C}$.

ASSUMPTION A1. *Let $\{AB: UE\}$ and $\{A_1B_1: UE_1\}$ be harmonic sets on two secants intersecting in the exterior point U. Then $ABUE \overline{\wedge} A_1B_1UE_1$ and $ABUE \overline{\wedge} B_1A_1UE_1$.*

Remark. Assumption A1 is a consequence of Theorem 1.3 (d). In a Desarguesian plane, sets harmonic with respect to $\mathfrak{C}$ are harmonic sets in the ordinary sense, and Assumption A1 is satisfied. Much of what we say about ovals applies as well to a curve $\mathfrak{C}$ in an infinite plane provided that $\mathfrak{C}$ has the following properties: (1) no three points of $\mathfrak{C}$ are collinear; (2) no three tangents to $\mathfrak{C}$ are concurrent. (At each point of $\mathfrak{C}$ there must be a tangent which contains no other points of $\mathfrak{C}$.) Now in the infinite case, we have a very natural example of a plane in which A1 is satisfied. We refer to the well known example **(8)** of a non-Desarguesian plane constructed by distorting the interior of a conic in a Desarguesian plane.

DEFINITION. If Assumption A1 is satisfied for every exterior point of an oval $\mathfrak{C}$, we shall call $\mathfrak{C}$ an *harmonic oval.*

THEOREM 2.1. *If U and V are exterior points of the harmonic oval $\mathfrak{C}$, and V is on the polar of U, then U is on the polar of V.*

Proof. Let A be a point on $\mathfrak{C}$, not on the polars of either U or V. Then the line UA is not a tangent and intersects $\mathfrak{C}$ in another point B. Similarly, let the line VA intersect $\mathfrak{C}$ in a point A_1. Now A_1 cannot be on the polar of U, else the line VA is the polar of U, contrary to the choice of A. Hence the line UA_1 intersects $\mathfrak{C}$ in another point B_1. Let $\{AB: UE\}$ and $\{A_1B_1: UE_1\}$ be harmonic sets. By Assumption A1, the lines AA_1, BB_1, and EE_1 are concurrent. But EE_1 is the polar of U. Therefore, AA_1 and EE_1 intersect in V, since V is on the polar of U by hypothesis. Hence BB_1 goes through V.

Now let $\{AA_1: VD\}$ and $\{BB_1: VF\}$ be harmonic sets. Again, by Assumption A1, AB, A_1B_1, and DF are concurrent. But DF is the polar of V, while AB and A_1B_1 intersect in U. Hence U is on the polar of V.

LEMMA 2.1. *Let U and V be conjugate exterior points of an harmonic oval $\mathfrak{C}$, (i.e., U and V are on each other's polars) then* (a) *the line UV is a secant if $n \equiv 1 \pmod 4$,* (b) *the line UV is an exterior line if $n \equiv -1 \pmod 4$.*

Proof. Let A be an absolute point not on the polars of U or V and not on the line UV. As in the proof of Theorem 2.1, a set of four absolute points AA_1BB_1 is determined such that U and V are diagonal points of the quadrangle AA_1BB_1. The $n+1$ points of $\mathfrak{C}$ occur in sets of four points, omitting (1) two points each on the polars of U and V and (2) the absolute points, if any, on the line UV. If UV is a secant, then 4 must divide $n-1=(n+1)-2$. If UV is an exterior line, then 4 must divide $n+1$.

THEOREM 2.2. *If $n \equiv 1 \pmod 4$, the fourth harmonic point is always exterior, while if $n \equiv -1 \pmod 4$, the fourth harmonic point is always interior.*

Proof. Consider the lines connecting an exterior point U to the exterior points on its polar u. If $n \equiv 1 \pmod 4$, every one of these lines is a secant. There are $\frac{1}{2}(n-1)$ exterior points on u and $\frac{1}{2}(n-1)$ secants through U. Thus all of the secants are accounted for and in each one of these cases the fourth harmonic point is exterior. If $n \equiv -1 \pmod 4$, every one of these lines is an exterior line. All of the exterior lines are accounted for and the remaining non-absolute lines through U must be secants which intersect u in interior points.

COROLLARY. *Given a collineation which* (a) *carries the oval $\mathfrak{C}$ into itself* (b) *is not of order* 2, (c) *leaves fixed a proper subplane with n_1+1 points on each line,* (d) *leaves at least one point of $\mathfrak{C}$ fixed, then $n \equiv n_1 \pmod 4$.*

Proof. The fixed points of $\mathfrak{C}$ form an oval in the subplane. If n is not equal to 2, a secant is fixed if and only if its absolute points are fixed. A fixed exterior point of $\mathfrak{C}$ is an exterior point of the oval and is the pole of a fixed secant which will also be a secant of the oval $\mathfrak{C}_1$. Sets harmonic with respect to $\mathfrak{C}_1$

are harmonic with respect to $\mathfrak{C}$, and a point of the subplane is respectively interior, exterior or an absolute point of $\mathfrak{C}_1$ if and only if it is an interior, exterior or absolute point of $\mathfrak{C}$, as the case may be. Thus if the fourth harmonic point is always exterior or always interior in one case, it must be so in the other case also.

DEFINITION. A triangle consisting of an exterior point U and two absolute points A and B such that U is the pole of the secant AB will be called a *fundamental triangle.*

THEOREM 2.3 *Let A, B, A_1, B_1 be four points on the harmonic oval $\mathfrak{C}$. Let C and C_1 respectively be the poles of the secants AB and A_1B_1. Then if AB and A_1B_1 intersect in an exterior point U, the fundamental triangles ABC and $A_1B_1C_1$ are perspective from a point. If AA_1 and BB_1 also intersect in an exterior point, the triangles are also perspective from a line.*

Proof. Since U is on the polar AB of the exterior point C, C is on the polar of U. Similarly, C_1 is on the polar of U. Let E and E_1 be the respective points of intersection of CC_1 with AB and A_1B_1. Then $\{AB: UE\}$ and $\{A_1B_1: UE_1\}$ are harmonic sets. Therefore, AA_1, BB_1 and $EE_1 = CC_1$ are concurrent in a point V and the triangles are perspective from V. Let D and D_1 be the respective poles of the secants AA_1 and BB_1. Then the fundamental triangles AA_1D and BB_1D_1 are perspective from the line CC_1. Similarly, if V is an exterior point, the triangles ABC and $A_1B_1C_1$ are perspective from the line DF.

THEOREM 2.4 *Let A, B, C, A_1, B_1, C_1 be six distinct absolute points of an harmonic oval $\mathfrak{C}$. If AA_1, BB_1, and CC_1 intersect in an exterior point V, and if P, Q, R are the respective points of intersection of the pairs of lines AB_1, A_1B; BC_1, B_1C; AC_1, A_1C; then P, Q, and R are collinear.*

Proof. Let $\{AA_1: VE\}$ and $\{BB_1: VD\}$ be harmonic sets. Then $AA_1VE \overline{\wedge} B_1BVD$, i.e., lines AB_1, A_1B, and ED are concurrent, where the line ED is the polar of V. Thus P is on the polar of V. Similarly, Q and R are on the polar of V.

COROLLARY. *If two harmonic ovals have five points A, B, C, A_1, and B_1 in common, and if AA_1 and BB_1 intersect in a point U exterior to both, then either the line UC is a common tangent or contains another point C_1 common to both.*

Remark. Qvist **(10)** has shown that if two ovals have half of their points in common, they are identical. Our Corollary suggests that five points may determine an harmonic oval if there are no proper subplanes.

THEOREM 2.5. *Let A, B, C be three points on a secant, and let A^2, B^2, C^2 be three points on another secant, where A, B, A^2, B^2 are absolute points of the harmonic oval $\mathfrak{C}$ and the two secants intersect in an exterior point. Then A, B, and C can be carried into A^2, B^2, C^2 by the product of at most two perspectivities on $\mathfrak{C}$.*

Proof. Suppose that CA^2 is not a tangent. Then the line CA^2 contains another absolute point B_1. Let BB_1 and AA^2 intersect in O_1. Then

$$ABC \overset{\mathbb{C}}{\wedge} A^2B_1C \overset{\mathbb{C}}{\wedge} A^2B^2C^2.$$

If CA^2 is a tangent, a similar argument can be made using one of the lines CB^2, C^2A, or C^2B unless all of these lines are also tangents. In this case, C is the pole of A^2B^2, while C^2 is the pole of AB. Then, by Theorem 2.3, the fundamental triangles A^2B^2C and ABC^2 are perspective from some point O, and $ABC \overset{\mathbb{C}}{\wedge} A^2B^2C^2$.

DEFINITION. A point to point transformation from a secant onto a secant will be called *projective with respect* to $\mathbb{C}$ (or merely, *projective*) if it carries sets harmonic with respect to the harmonic oval $\mathbb{C}$ into sets harmonic with respect to $\mathbb{C}$.

THEOREM 2.6. *A collineation which carries $\mathbb{C}$ into itself induces a projective mapping between the points of a secant and its image.*

DEFINITION. If $n \equiv 1 \pmod 4$, a triangle UVW such that each of the exterior points U, V, W is the pole with respect to the oval $\mathbb{C}$ of the opposite side of the triangle will be called a *self polar triangle with respect to* $\mathbb{C}$. If V and W are on the line l, the triangle will be said to be a *self polar triangle on* l.

THEOREM 2.7. *Let $n \equiv 1 \pmod 4$. Let l and l_1 be two secants of the oval $\mathbb{C}$, intersecting in an exterior point. Then a perspectivity from l onto l_1 will be projective with respect to $\mathbb{C}$ if and only if self polar triangles on l are perspective to self polar triangles on l_1. If the center of perspectivity is an exterior point V, the triangles are also perspective from the polar of V.*

Proof. A perspectivity on $\mathbb{C}$ carries absolute points into absolute points. If C is the pole of l and C_1 is the pole of l_1, it follows from Theorem 2.3 that the center of perspectivity is on the line CC_1. If CDE is a self polar triangle, then $\{AB: DE\}$ is a harmonic set, where A and B are the absolute points on l. The image $\{A_1B_1: D_1E_1\}$ of $\{AB: DE\}$ will be a harmonic set if and only if D_1 and E_1 are on each other's polars, i.e., $C_1D_1E_1$ is a self polar triangle. The last part of the Theorem follows from Theorem 2.1.

Assumption A1 does not tell us much about interior points. If $n \equiv -1 \pmod 4$, it is natural to extend this assumption in the following manner:

ASSUMPTION A2. *If $\{AB: UE\}$ and $\{A_1B_1: U_1E\}$ are harmonic sets with respect to the oval $\mathbb{C}$, where the point E is an interior point, then*

$$ABUE \overset{\mathbb{C}}{\wedge} A_1B_1U_1E, \quad ABUE \overset{\mathbb{C}}{\wedge} B_1A_1U_1E.$$

We shall also make use of another Assumption. (Note Theorem 2.3.)

ASSUMPTION B: *Given two fundamental triangles ABC and $A_1B_1C_1$, where A, B, A_1, B_1 are absolute points of the oval $\mathbb{C}$, then the triangles are perspective from the intersection of the secants AA_1, and BB_1 and also from the point of intersection of the secants AB_1 and A_1B.*

LEMMA 2.2. *If* $n \equiv -1 \pmod 4$, *and Assumptions* A1, A2, B *are satisfied for the oval* $\mathfrak{C}$, *then the poles of the secants through each interior point* O *are collinear.*

Proof. Given any secant, corresponding to each exterior point on that secant there is an interior point which is its harmonic conjugate. Since the number of exterior points on a secant is equal to the number of interior points, it is likewise true that for every interior point on a secant there is an exterior point on the secant which is its harmonic conjugate. Let AB be a secant through O, and let U be an exterior point such that $\{AB: UO\}$ is a harmonic set. Let U_1 be the pole of the secant AB. We shall show that the pole of every secant through O lies on the line UU_1.

Let CD be any other secant through O and let $\{CD: VO\}$ be a harmonic set. Let V_1 be the pole of CD. Then, by Assumption A2, the lines AC, BD, and UV are concurrent. Thus the line UV connects the point of intersection of the lines AC and BD with the point of intersection of AD and BC. Now consider the fundamental triangles ABU_1 and CDV_1. The two sets of lines AC, BD, U_1V_1 and AD, BC, U_1V_1 are respectively concurrent. The lines UV and U_1V_1 must be the same, i.e., V_1 lies on the line UU_1.

THEOREM 2.8. *If* $n \equiv -1 \pmod 4$, *and Assumptions* A1, A2, *and* B *are satisfied for the oval* $\mathfrak{C}$, *then* $\mathfrak{C}$ *determines a polarity.*

Proof. The polar of an interior point O can be taken as the line containing the poles of the secants through O. We have completely determined a correspondence between points and lines. This correspondence will be a polarity if, for any two points U and V such that V is on the polar of U, then U is on the polar of V. Theorem 2.1 takes care of the case where both U and V are exterior. The case where U is exterior and V is interior is taken care of by the way in which we defined the polar of an interior point. U interior and V exterior can be handled in a similar manner. The case where both U and V are interior is readily handled by an argument similar to those used in Theorems 2.1 and 2.3.

3. Conics in projective planes. As remarked before, Baer **(1)** has shown that the absolute points of a polarity form an oval unless n is even or a square. In §2, we had a sort of quasi-polarity but were restricted by the fact that the relation of pole and polar was restricted to exterior points and secants.

DEFINITION. A set of $n + 1$ points will be called a *quasi-conic* if (1) no three of them are collinear, and (2) they are the absolute points of a polarity.

DEFINITION. A set of four collinear points $\{AB: CD\}$ will be said to be *harmonic with respect to the quasi-conic* $\mathfrak{C}$ if (1) A and B belong to $\mathfrak{C}$, (2) C and D are conjugate points of the polarity.

ASSUMPTION A. *Let* $\{AB: UE\}$ *and* $\{A_1B_1: UE_1\}$ *be harmonic sets with respect to the quasi-conic* $\mathfrak{C}$. *Then* $ABUE \overline{\wedge} A_1B_1UE_1$ *and* $ABUE \overline{\wedge} B_1A_1UE_1$.

DEFINITION. If Assumption A is satisfied for every pair of sets harmonic with respect to the quasi-conic $\mathfrak{C}$, then $\mathfrak{C}$ will be said to be a *conic*.

We will be able to put some of the theorems of §2 in a simpler form and to obtain some new ones. Many of the proofs are similar to those of §2, so we shall give the proofs only for those theorems which have no counterpart in §2. We shall number the theorems to facilitate cross-reference—e.g., Theorem 3.3 will be analogous to Theorem 2.3. Theorems which carry over unchanged will be omitted. This will cause some gaps in the numbering of theorems.

THEOREM 3.3. *For a quasi-conic* $\mathfrak{C}$, *the following propositions are equivalent:*

(a) *Assumption* A.

(b) *Given two fundamental triangles* ABC *and* $A_1B_1C_1$, *where* A, B, A_1, *and* B_1 *are absolute points, the triangles are perspective from the intersection of the secants* AA_1, *and* BB_1, *and also from the point of intersection of the secants* AB_1 *and* A_1B.

(c) *The triangles in* (b) *are perspective from the polars of the points which are the centers of perspectivity in* (b).

(d) *If the points of a complete quadrangle are all absolute points, then the diagonal points form a self polar triangle—i.e., each point is the pole of the opposite side.*

THEOREM 3.4. *Let* A, B, C, A_1, B_1, C_1 *be six distinct absolute points of a conic. If* AA_1, BB_1, *and* CC_1 *intersect in a point* V, *and if* P, Q, *and* R *are the respective points of intersection of the pairs of lines* AB_1, A_1B; BC_1, B_1C; *and* AC_1, A_1C; *then* P, Q *and* R *are collinear.*

COROLLARY. *If two conics have five points in common, then either they have a common tangent at one of these points or there is another point common to both.*

THEOREM 3.5. *Let* A, B, C *be three points on a secant, and let* A^2, B^2, C^2 *be three points on another secant, where* A, B, A^2, B^2 *are absolute points of a conic* $\mathfrak{C}$. *Then* A, B, C *can be carried into* A^2, B^2, C^2 *by the product of at most two perspectivities on* $\mathfrak{C}$.

THEOREM 3.5.1. *Let* $\{AB: UT\}$ *be a harmonic set with respect to the conic* $\mathfrak{C}$, *where* U *is an exterior point. Then there is a complete quadrangle such that* U *and* T *are diagonal points, while* A *and* B *lie on the other two sides of the quadrangle—i.e.,* $\{AB: UT\}$ *forms a harmonic set in the sense of the usual definition.*

Proof. Let U_1 be the pole of the secant AB, so that U_1A and U_1B are tangents at A and B respectively. Let UA_1 and UB_1 be tangents from U, tangent at the absolute points A_1 and B_1. Since U is on AB, U and U_1 are conjugate so that U_1 lies on A_1B_1.

Let UA_1 and U_1A intersect in X,
UA_1 and U_1B intersect in Y,
UB_1 and U_1A intersect in Z,
UB_1 and U_1B intersect in W.

Then U and U_1 are two of the diagonal points of the complete quadrangle $XYZW$. We shall show that T is the third diagonal point, i.e., XW and YZ intersect in T.

Consider the fundamental triangles UA_1B_1 and U_1AB. By Theorem 3.3, they are perspective from the intersection of AB_1 and A_1B, and also from the polar of this point. The corresponding pairs of sides are UA_1, U_1B; UB_1, U_1A; and AB, A_1B_1. The points of intersection Y, Z, and T are therefore collinear. Similarly, X, W, and T are collinear. Thus T is the third diagonal point. Since A and B lie on XZ and YW respectively, this completes our proof.

THEOREM 3.7. *A perspectivity on a conic $\mathfrak{C}$ from a secant l onto a secant l_1 will be projective with respect to $\mathfrak{C}$ if and only if self polar triangles on l are perspective to self polar triangles on l_1.*

THEOREM 3.7.1. *If perspectivities on $\mathfrak{C}$ are projective with respect to the conic $\mathfrak{C}$, then any set harmonic with respect to the conic $\mathfrak{C}$ can be carried into any set harmonic with respect to $\mathfrak{C}$ by a product of perspectivities on C.*

The proof follows from Theorem 3.5.

4. Ovals in transitive planes. A cyclic plane is a plane which possesses a cyclic group of collineations, transitive on the points of the plane. Zappa **(12)** has studied a more general class of transitive planes. The author is indebted to Professor B. Segre for suggesting that some of the results of this part might be applied to the more general class of transitive planes.

A cyclic plane can be represented in the following manner: the points can be taken as a complete set of residues mod $N = n^2 + n + 1$. Let $a_0, a_1, \ldots, a_n$ form a difference set mod N. That is, the set $\{a_i - a_j\}, i \neq j$ $(i, j = 0, 1, \ldots, n)$ contains each residue mod N exactly once. For any fixed s, the set $\{a_i - s\}$ $(i = 0, 1, \ldots, n)$ will also be a difference set. The sets $\{a_i - s\}$ can be taken as the lines of a projective plane and will be denoted by l_s. In discussing cyclic planes, equations will denote congruences mod N.

There are two known collineations in a cyclic plane: (1) the mapping $x \to x + s$, (2) for certain choices of m, the mapping $x \to mx$ is a collineation. Such values of m are called multipliers. Hall **(5)** has showed that all factors of n are multipliers. There is at least one line which is left invariant by all multipliers. Take l_0 to be such an invariant line.

Now the mapping $x \to l_x$ forms a natural polarity **(5)**. Two points x and y are conjugate points of the polarity if, for some i, $x + y = a_i$, since then $x \in l_y$ and $y \in l_x$. The absolute points of the polarity will be the points x such that $2x = a_i$ $(i = 0, 1, \ldots, n)$ and will be the points of a quasi-conic. (If n is even, 2 is a multiplier and the absolute points all lie on l_0. See **(1**, p. 83**)**).

More generally, $x \to l_{x+s}$ forms a polarity in which x and y are conjugate if $x + y = a_i - 2s$. The absolute points are the points x such that $2x = a_i - 2s$ or $x = \frac{1}{2}a_i - s$. Let $\mathfrak{C}_s$ denote the set $\frac{1}{2}a_i - s$ $(i = 0, 1, \ldots, n)$. $\mathfrak{C}_s$ then is a quasi-conic for each s, $s = 0, 1, \ldots, N$. Now, the set $\mathfrak{C}_0$ is a difference set mod N. The sets $\mathfrak{C}_s$ form a projective plane π_1 in the same way that the sets l_x formed the plane π. Moreover, $\mathfrak{C}_0$ is invariant under the collineation $x \to mx$, m a multiplier.

THEOREM 4.1. *Given any three collinear points A, B, C, there exists a unique point D such that the set $\{AB: CD\}$ is harmonic with respect to one of the quasi-conics $\mathfrak{C}_s$.*

Proof. Since the conics $\mathfrak{C}_s$ form the lines of a finite projective plane π_1, there is exactly one quasi-conic $\mathfrak{C}_s$ such that A and B are points on $\mathfrak{C}_s$. With respect to $\mathfrak{C}_s$, the point C has exactly one conjugate point D on the line containing A, B, and C.

Next, consider the correlation $x \to l_{mx}$, where m is a multiplier. (From here on, to avoid continual use of subscripts, we shall use small letters near the beginning of the alphabet to denote elements of l_0.) A point x will be an absolute point of the correlation if it lies on its image line $\bar{l}_{mx}$ i.e., if $x = a - mx$ for some $a \in l_0$. The absolute points of the correlation are thus the solutions of the equations $(m + 1)x = a$, $a \in l_0$. Now if $m + 1$ is a divisor of zero, then m is congruent to -1 modulo some factor of N. By Mann's theorem **(7)** on multipliers of even order, n must then be a square. If n is not a square, we have exactly $n + 1$ absolute points $x = a(m + 1)^{-1}$, where a may take on any of the values $a_0, a_1, \ldots, a_n$.

If $a(m + 1)^{-1}$ is an absolute point, its image line $\bar{l}_{ma(m+1)^{-1}} = \bar{l}_{b(m+1)^{-1}}$ is an absolute line, where $b = ma \in l_0$.

We know that each absolute line contains at least one absolute point. Suppose that $\bar{l}_{a(m+1)^{-1}}$ contains the absolute point $b(m + 1)^{-1}$. Then, for some $c \in \bar{l}_0$,

$$b(m + 1)^{-1} = c - a(m + 1)^{-1}.$$

Multiplying through by $m + 1$ and transposing, we get

$$b - cm = c - a$$

where $a, b, c, cm \in \bar{l}_0$. Since $\bar{l}_0$ is a difference set, this implies either

$$\begin{cases} b = cm \\ c = a \end{cases} \text{ or } \begin{cases} b = c \\ cm = a \end{cases}$$

Therefore, either $b = am$ or $b = am^{-1}$. Conversely, if $b = am$ or am^{-1}, then the absolute point $b(m + 1)^{-1}$ belongs to the absolute line $\bar{l}_{k(m+1)^{-1}}$.

If $m = 1$, we have the polarity $x \to l_x$ and each absolute line contains one absolute point. Even if $m \neq 1$, we may have $ma = a$ for certain $a \in \bar{l}_0$ if a is a fixed point under the collineation $x \to mx$. In all other cases, each absolute line $\bar{l}_{a(m+1)^{-1}}$ contains the two absolute points $am^{-1}(m + 1)^{-1}$ and $am(m + 1)^{-1}$ i.e., the image and inverse image of the absolute line. Hence

LEMMA 4.1. *Each absoltue line of the correlation contains at most two absolute points.*

LEMMA 4.2. *If* (a) m *is a multiplier,* (b) n *is not a square,* (c) $a \in l_0$, $ma \neq a$ *then* $(m+1)a$ *does not belong to* l_0.

Proof. An element of l_0 multiplied by a multiplier gives an element of l_0. If $(m+1)a \in l_0$, then $m(m+1)a$ and $m^2(m+1)a$ also belong to l_0. If a, ma, m^2a are distinct, we will be led to a contradiction of Lemma 4.1. If $m^2a = a$, but $ma \neq a$, then $(m-1)a \neq 0$, $(m-1)(m+1)a = 0$ so that $m+1$ is a divisor of zero which implies that n is a square.

THEOREM 4.2. *If* n *is not a square and if* m *is a multiplier* $\neq 1$, *then* $m+1$ *is not a multiplier.*

Proof. If $m+1$ is a multiplier and $a \in l_0$ then $(m+1)a \in l_0$. By Lemma 4.2, this can only happen if $ma = a$ for every $a \in l_0$. Let $(m-1, N) = N_1$. Let $N_2 = N/N_1$. If $(m-1)a = 0$, then a must contain N_2 as a factor. If this be true for every $a \in l_0$ then all differences between elements of l_0 are multiples of N_2 and l_0 cannot be a difference set.

COROLLARY. *If* m_1 *and* m_2 *are multipliers, then* $m_1 + m_2$ *is not a multiplier.*

Remark. These are strengthened versions of Mann's theorem 3 and Corollary 1 in reference **(7)**. The corresponding proposition for $m = 1$ is as follows: "2 is a multiplier if and only if n is even." This has already been noted by Hall **(5)**.

Of particular interest is the case where $m = n$. Now the set $\{a_i\}$ of points $\in l_0$ is the same as the set $\{n^{-1}a_i\}$; the set $a_i(n+1)^{-1}$ of absolute points is the same as the set $\{-a_i\}$ since $n(n+1) \equiv -1 \pmod{N = n^2+n+1}$.

THEOREM 4.3. *The absolute points of the correlation* $x \to l_{nx}$ *form an oval* $\mathfrak{C}^*$.

Proof. We have already proved that there are $n+1$ absolute points. Now if a and $b \in l_0$ then the line l_{a+b} contains $-a = b - (a+b)$ and $-b = a - (a+b)$. Suppose that $-a$, $-b$, $-c$ are collinear, where a, b, $c \in l_0$. Then $l_{a+b} = l_{a+c} = l_{b+c}$ so that $a+b = a+c = b+c$ and a, b, and c are not distinct.

THEOREM 4.4. *Consider the collineation* $x \to x + a - b$, *which carries* $-a$ *into* $-b$, *where* $a, b \in l_0$. *The intersection of each line through* $-a$ *and its image line through* $-b$ *is a point of* $\mathfrak{C}^*$.

Proof. The lines through $-a$ can be written in the form l_{a+c} where $c \in l_0$. (If $c = a$, we have the tangent line at $-a$.) The image line under the collineation is the line $l_{a+c-a+b} = l_{c+b}$ which intersects l_{a+c} in the point $-c$.

We have earlier remarked that in any plane with a polarity, self polar triangles which are perspective from a point are also perspective from a line. We shall proceed to demonstrate pairs of self polar triangles which are perspective from a point.

LEMMA 4.3. *If a belongs to l_0 but is not equal to 0, $N/3$ or $2N/3$, then $-a$, $-na$, $-n^2a$ are the vertices of a triangle which is self polar under the polarity $x \to l_x$.*

Proof. Since $a + na = -n^2a$, $-n^2a$ is the pole of l_{a+na}, the line through $-a$ and $-na$. Similarly, $-a$ and $-na$ are the poles of the other two sides. $-a$, $-na$, $-n^2a$ are distinct unless $a = 0$, $N/3$ or $2N/3$. We have already seen that they are not collinear (Theorem 4.3).

THEOREM 4.5. *If n is odd and not divisible by 3, then there exist pairs of self polar triangles which are perspective from 0 and therefore from l_0.*

Proof. 0 does not belong to l_0 unless n is divisible by 3 **(3)**. If $n \equiv 1 \pmod 3$, $N \equiv 0 \pmod 3$ and there may be three lines invariant under all multipliers. If so, two of these will contain 0, but we can still take l_0 as a line which does not contain 0.) Likewise, 0 does not belong to the oval $\mathfrak{C}^*$. Consider a secant intersecting $\mathfrak{C}^*$ in the points $-a$, $-b$ where a, $b \in l_0$. Then the triangles $-a$, $-na$, $-n^2a$ and $-b$, $-nb$, $-n^2b$ are the desired triangles provided $a \neq N/3$ or $2N/3$.

Suppose that a projective plane π admits a group Σ of collineations which (a) is transitive on the points and lines of π, and (b) the only element of Σ which leaves fixed a given point 0 is the identity. Zappa **(12)** calls such a plane a regular transitive plane. We shall need to require in addition that (c) Σ is abelian.

Associated with each point x of π will be the element σ_x of Σ which carries 0 into x. Let l_0 denote one of the lines of π. Let l_x be the image of l_0 under the collineation σ_x^{-1}. Then if $x \in l_y$, $\sigma_x = \sigma_a\sigma_y^{-1}$ for some $a \in l_0$. If Σ is abelian, $\sigma_y = \sigma_a\sigma_x^{-1}$ and $y \in l_x$. Hence the mapping $x \to l_x$ is a polarity. The absolute points are the points x such that $\sigma_x^2 = \sigma_a$ for some $a \in l_0$.

Let $-x$ denote the image of 0 under σ_x^{-1}; let $x + y$ be the image of 0 under the collineation $\sigma_x\sigma_y$. If a and $b \in l_0$ then l_{a+b} contains $-a$ and $-b$. As in Theorem 4.3, if $a_0, a_1, \ldots, a_n$ are the points of l_0 it follows that $-a_0, -a_1, \ldots, -a_n$ form an oval $\mathfrak{C}^*$. The analogue of Theorem 4.4 goes through. Let μ be a collineation (not the identity) which leaves 0 and l_0 fixed. Then μ will also carry $\mathfrak{C}^*$ into itself. Denote the image of x under μ by $\mu(x)$. Then the mapping $x \to l_{\mu(x)}$ is a correlation. If x is an absolute point of this collineation, then $\sigma_x = \sigma_a\sigma_{\mu(x)}^{-1}$ for some $a \in l_0$ and $\sigma_x\sigma_{\mu(x)} = \sigma_a$. As in Lemma 4.2, it follows that we cannot have points a and $b \in l_0$ such that $\sigma_b\sigma_{\mu(b)} = \sigma_a$ unless $\mu(b) = b$ or $\mu^2(b) = b$.

If μ is such that, for every point x,

$$\sigma_x\sigma_{\mu(x)}\sigma_{\mu^2(x)} = \sigma_0$$

then the absolute points of the correlation $x \to l_{\mu(x)}$ satisfy the equation

$$\sigma_a^{-1} = \sigma_{\mu^2(x)}$$

i.e., $-a = \mu^2(x)$, $\mu^{-2}(-a) = x$ where $a \in l_0$, so that $-a$ and $\mu^{-2}(-a)$ belong

to $\mathfrak{C}^*$. Thus $\mathfrak{C}^*$ consists of the absolute points of the correlation $x \to l_{\mu(x)}$. For the analogue to Theorem 4.5 to go through, μ must be of order 3.

5. The existence of Fano's configuration. In this part, we are concerned with the case where n is even. Here we shall not be concerned with ovals, which do not have the same properties as when n is odd. However, Theorem 5.1 is, in a sense, the counterpart of Theorem 4.5. Recall that, if n is even, and not a square, the absolute points of a polarity are the points of a line.

THEOREM 5.1. *Let the absolute points of a polarity be the points on a line o, and let O be the pole of o. Let A, B, C be the vertices of a self polar triangle, where A, B, C are not absolute points and are all different from O. Then the diagonal points of the complete quadrangle with vertices O, A, B, C are all on o.*

Proof. First we show that the lines OA and BC must intersect in a point of o. Let D denote the intersection of the line BC with o. Since A is the pole of BC, A is conjugate to D. Therefore, the polar of D goes through O and A. But D is an absolute point, hence the polar of D goes through D, i.e., the line OA contains D. Thus D is a diagonal point of the quadrangle $OABC$. Similarly, the other two diagonal points lie on o.

Remark. Note that Theorem 5.1 does not require that the plane be finite.

THEOREM 5.2. *In a cyclic plane, if n is even and not a square, if $a \in l_0$, then 0, $-a$, $-2a$, $-4a$ are the vertices of a complete quadrangle for which the diagonal points are collinear.*

Proof. For a cyclic plane, n cannot be divisible **(5)** by both 2 and 3. Now $N \equiv 0 \pmod 3$ if and only if $n \equiv 1 \pmod 3$. If $N \equiv 0 \pmod 3$, the multiplier 2 is even order and n must be a square. Thus $n \equiv 1 \pmod 3$; $0 \in l_0$ and 3 is not a divisor of zero. We conclude that 0, $-a$, $-2a$, $-4a$ are distinct. Theorem 4.3 applies as well when n is even, so $-a$, $-2a$, $-4a$ are not collinear. It may be shown by similar methods that no three of the points 0, $-a$, $-2a$, $-4a$ are collinear.

Now, since 2 is a multiplier, l_0 contains a, $2a$, $4a$, $8a$. ($8a$ does not necessarily differ from a). Hence l_{2a} contains $2a - 2a = 0$, $a - 2a = -a$, $4a - 2a = 2a$. Similarly, l_{4a} contains $-4a$, $-2a$, and $2a$. Thus the lines 0, $-a$ and $-4a$, $-2a$ intersect in $2a$, i.e., $2a$ is one of the diagonal points. Similarly, the lines 0, $-2a$ and $-a$, $-4a$ intersect in the point $-3a$. Now l_{2a} is the line 0, $-4a$ while l_{3a} is the line $-a$, $-2a$. If we denote by x the intersection of l_{4a} and l_{3a} then for some c, $d \in l_0$,

$$c - 8a = d - 3a = x.$$

Then c and d are the elements of l_0 such that

$$c - d = 8a - 3a = 5a.$$

To show that the diagonal points $2a$, $-3a$, x are collinear, consider the line l_{d+3a}. This line contains

$$d - (d + 3a) = -3a, \quad c - (d + 3a) = 5a + d - (d + 3a) = 2a \quad \text{and}$$
$$2d - (d + 3a) = x.$$

THEOREM 5.3. *Suppose that a projective plane admits a collineation of order two such that every point on a line l is fixed and every line through a point $P \in l$ is fixed. If A and B are two non-fixed points not collinear with P, if A_1 and B_1 are the respective images of A and B, then the diagonal points of the quadrangle ABA_1B_1 all lie on l.*

Proof. A and A_1 are collinear with P; similarly, B and B_1 are collinear with P. Hence P is one of the diagonal points. The line A_1B_1 is the image of AB; if Q is the intersection of AB with l, A_1B_1 must also go through Q. Similarly, AB_1 and A_1B must intersect in a point on l, since $AB_1 \rightleftarrows A_1B$.

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ON THE COORDINATIZATION THEOREM OF J. VON NEUMANN

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1. Introduction. A classical theorem of n-dimensional projective geometry asserts: if Desargues' theorem holds (in particular, if $n \geqslant 3$) then the points of the geometry can be assigned homogeneous coordinates $(\alpha^1, \ldots, \alpha^{n+1})$ with α^i not all zero in a suitable division ring $\mathfrak{R}$ (**2**, p. 104; **3**, Theorems 10, 11, ex. 19, p. 204). A deep generalization by John von Neumann (**5**, Theorem 14.1, p. 141) proved such a theorem for every complemented modular lattice possessing a homogeneous basis of $n + 1$ elements, $n \geqslant 3$. (The von Neumann theorem provides coordinates for all the elements in the lattice, so that in the special case of the projective geometry, coordinates are provided not merely for points, but for all points, lines, planes, etc. In the general complemented modular lattice there need not be "points".) For the general complemented modular lattice the ring $\mathfrak{R}$ may be an arbitrary regular ring with unit.[1]

A precise statement of the von Neumann theorem is as follows:

Let L be a complemented modular lattice possessing a homogeneous basis[2] *$a_1, \ldots, a_n$, $n \geqslant 4$, and let L_{12} be the set of all inverses of a_2 with respect to $a_1 + a_2$. Then* (i) *addition and multiplication can be defined for the elements of L_{12} in such a way that L_{12} becomes a regular ring $\mathfrak{R}$ and the sub-lattice $L(a_1)$, consisting of all $x \leqslant a_1$, is isomorphic to the lattice of all left principal ideals of $\mathfrak{R}$, and* (ii) *L is lattice-isomorphic to (coordinatized by) the lattice of all left principal ideals of $\mathfrak{R}_n$.*

Here $\mathfrak{R}_n$ consists of all $n \times n$ matrices with elements in $\mathfrak{R}$ and, as shown by von Neumann, is regular along with $\mathfrak{R}$. It was shown by von Neumann that (ii) above is equivalent to the more classical form of coordinatization, (ii)′ L is lattice-isomorphic to the lattice of $\mathfrak{R}$-left modules of finite span in the space V of vectors $v = (\alpha^1, \ldots, \alpha^n)$ with all α^i in $\mathfrak{R}$.

The definitions of addition and multiplication as originally given by von

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[1] Von Neumann (**5**, Chapter 2) calls a ring *regular* if for each x there is a y satisfying $xyx = x$.

[2] Lattice elements $a_1, \ldots, a_m$ are called *independent* if: $(a_1 + \ldots + a_{i-1})a_i = 0$ for $i = 2, \ldots, m$. A *relative complement* or *inverse* of x in z is any element $[z - x]$ for which $x \oplus [z - x] = z$ ($\oplus$ is used to indicate that the addends are independent). Lattice elements x and y are called *perspective*, written $x \sim y$, if there exists any z, called an *axis of perspectivity*, such that $x \oplus z = y \oplus z$. Independent elements $a_1, \ldots, a_n$ with $a_1 + \ldots + a_n = 1$ are said to form a basis for L; if also $a_i \sim a_j$ for each i, j, the basis is called *homogeneous*; a homogeneous basis $a_1, \ldots, a_n$ together with axes of perspectivity c_{ij} with the properties:

$$c_{ij} \oplus a_i = c_{ij} \oplus a_j;\ c_{ij} = c_{ji};\ c_{ii} = 0;\ (c_{ij} + c_{jk})(a_i + a_k) = c_{ik};$$

is called a *normalized frame* for L. The collection of all inverses of a_j in $a_i + a_j$ will be denoted by L_{ij}. (**1**, Sections 2.3, 2.4, 2.5, 4.1; **4**, Chapter 2; **5**, Definitions 3.1, 3.3, 5.2.)

Neumann for (i) above, were motivated in part by the classical constructions in projective geometry, but the verification that $\mathfrak{R}$ becomes a regular ring was based on a rather involved analysis. A simplified discussion of this part of von Neumann's theorem was given in **(1)**.

Von Neumann proved (ii) by showing that for any $m = 1, \ldots, n-1$, a given isomorphism between the lattice spanned by $a_1, \ldots, a_m$ and the lattice of left principal ideals of $\mathfrak{R}_m$ can be extended from m to $m+1$, with a special discussion for the extension from $m = n-1$ to $m = n$.

The object of this paper is to give a direct proof of (ii)′ (equivalent, as pointed out above, to (ii)) which does not use matrix rings or extension theorems.

As in **(5**, p. 32 and p. 64; **1**, §§4.1 and 4.3**)** we shall assume a fixed normalized frame for $L (n \geqslant 4)$ and $\mathfrak{R}$ will consist of all L-numbers $\alpha = \langle \alpha_{ij}; i, j = 1, \ldots, n, i \neq j \rangle$ for which each α_{ij} is in the corresponding L_{ij} and the identities hold: $(\alpha_{ij} + c_{jk})(a_i + a_k) = \alpha_{ik}$ and $(\alpha_{ij} + c_{ik})(a_j + a_k) = \alpha_{kj}$. V will denote the set of all vectors of length n with coordinates in $\mathfrak{R}$; a subscript l will denote either left ideal in $\mathfrak{R}$ or $\mathfrak{R}$-left module in V.

We shall give a rule which assigns to each element x in L a family of left modules in V. Then we shall show that all left modules assigned by this rule to x actually coincide and we shall verify that the rule sets up a (1, 1) order preserving correspondence (i.e., a lattice isomorphism) between L and the set of all left modules of finite span. This will establish the von Neumann coordinatization theorem.

2. Notation. We shall adopt the notation:

$$A^0 = 0; A^i = a_1 + \ldots + a_i, \qquad i = 1, \ldots, n;$$
$$A^i{}_j = a_1 + \ldots + a_{j-1} + a_{j+1} + \ldots + a_i, \qquad 0 \leqslant j \leqslant i \leqslant n.$$

A system of lattice elements $\langle b \rangle = \langle b_{ij}; i, j = 1, \ldots, n, i \neq j \rangle$, satisfying $b_{ij} \leqslant a_i + a_j$, will be called a *fraternal system* if the identities hold: $(b_{ij} + c_{jk})(a_i + a_k) = b_{ik}$; $(b_{ij} + c_{ik})(a_k + a_j) = b_{kj}$; if any, and hence each, of the b_{ij} is in the corresponding L_{ij}, the fraternal system coincides with an L-number. Von Neumann's work shows that for each $x \leqslant a_i + a_j$ there is one and only one fraternal system $\langle b \rangle$ with $b_{ij} = x$ **(5**, Lemma 6.1; **1**, §4.3**)**.

For future reference we list the following formulae:

(2.1) $\qquad (\alpha + \beta)_{ij} = [\alpha_{kj} + (\beta_{ij} + a_k)(c_{ik} + a_j)](a_i + a_j),$

(2.2) $\qquad (\alpha + \beta)_{ij} = [(\alpha_{ik} + a_j)(\beta_{ij} + a_k) + c_{jk}](a_i + a_j),$

(2.3) $\qquad (\alpha - \beta)_{kj} = [\alpha_{ij} + (a_k + \beta_{ij})(a_j + c_{ik})](a_k + a_j),$

(2.4) $\qquad (\alpha\beta)_{ij} = (\alpha_{ik} + \beta_{kj})(a_i + a_j).$

The formulae (2.1), (2.2), (2.3), and (2.4) may be obtained from **(3**, Chap. 7**)**; they are given explicitly in **(1)**, numbered as (4.9.1), (4.9.2), (4.9.3), and (4.10.1) respectively.

3. Reach and nullity. We associate with each α in $\mathfrak{R}$ a fraternal system which we shall call the *nullity* of α, defined as

$$\langle b \rangle \text{ with } b_{ij} = \alpha_{ij}\, a_i \text{ for all } i, j.$$

It follows from the fraternal character of the α_{ij} that the $\alpha_{ij}\, a_i$ do form a fraternal system; moreover $b_{ij} \leqslant a_i$ for all i, j, and hence is independent of j. We shall write $\alpha^0{}_i$ for this b_{ij}.

For each α in $\mathfrak{R}$, the *reach* of α is defined as in **(1, §4.13)** to be the fraternal system $\langle b \rangle$ with $b_{ij} = (\alpha_{ij} + a_i)a_j$, written as $\alpha^r{}_j$, identical with the $(\alpha)_j$ of von Neumann **(5, Definition 9.1)**. We shall use the following:

(3.1) For every $b \leqslant a_j$ and $d = [a_j - b]$ there is an idempotent e in $\mathfrak{R}$ with $e^r{}_j = b$ and[2] $(1 - e)^r{}_j = d$.

(3.2) For α, β in $\mathfrak{R}$ there is a γ satisfying $\gamma\alpha = \beta$ if and only if $\alpha^r{}_j \geqslant \beta^r{}_j$.

For proofs of (3.1) and (3.2) see **(5, Theorem 9.3, Lemma 9.1; 1, §4.13)**.

We shall now prove the following relations:

(3.3) $\alpha\beta = 0$ *if and only if* $\alpha^r{}_j \leqslant \beta^0{}_j$;

(3.4) $e^0{}_j = (1 - e)^r{}_j$ *for every idempotent* e;

(3.5) $e^0{}_j \oplus (1 - e)^0{}_j = e^0{}_j \oplus e^r{}_j = a_j$ *for every idempotent* e;

(3.6) $(\alpha - \beta)^0{}_i = (\alpha_{ij}\beta_{ij} + a_j)a_i$;

(3.7) $\alpha_{kj} \leqslant (\alpha - \beta)_{kj} + \beta^r{}_j$.

Proof of (3.3). $\alpha\beta = 0$ means

$$a_i = (\alpha_{ij} + \beta_{jk})(a_i + a_k).$$

Because of the indivisibility of inverses[3], this is equivalent to each of

$$a_i \leqslant (\alpha_{ij} + \beta_{jk})(a_i + a_k),$$

$$a_i \leqslant \alpha_{ij} + \beta_{jk},$$

(3.8) $$a_i + \alpha_{ij} \leqslant \alpha_{ij} + \beta_{jk},$$

(3.9) $$(a_i + \alpha_{ij})(a_j + a_k) \leqslant (\alpha_{ij} + \beta_{jk})(a_j + a_k).$$

(Add α_{ij} to both sides of (3.9) to derive (3.8).) Thus $\alpha\beta = 0$ is equivalent to each of:

$$(a_i + \alpha_{ij})a_j \leqslant \beta_{jk} + \alpha_{ij}(a_j + a_k),\ \alpha^r{}_j \leqslant \beta_{jk},\ \alpha^r{}_j \leqslant a_j\beta_{jk} = \beta^0{}_j.$$

Proof of (3.4). That $(1 - e)^r{}_j \leqslant e^0{}_j$ follows from (3.3). Let α be an L-number with $\alpha^r{}_j = [e^0{}_j - (1 - e)^r{}_j]$. Then $\alpha^r{}_j \leqslant e^0{}_j$; by (3.3) this implies $\alpha e = 0$, hence $\alpha(1 - e) = \alpha$, hence, by (3.2), $\alpha^r{}_j \leqslant (1 - e)^r{}_j$. Since $\alpha^r{}_j(1-e)^r{}_j = 0$ (from the definition of α) this implies $\alpha^r{}_j = 0$ and hence $e^0{}_j = (1 - e)^r{}_j$.

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[3]The indivisibility of inverses, which follows from the modular law, states: if y_1 and y_2 are both inverses of a in b and $y_1 \leqslant y_2$, then $y_1 = y_2$. Because of this, "points" as used in certain constructions in the classical theory of projective geometry may be replaced by "inverses", **(1, Section 2.4; 5, Lemma 3.3, p. 32)**.

Proof of (3.5). Because of (3.1) and (3.2), the correspondence $(\alpha)_i \leftrightarrow \alpha^r{}_j$ is a $(1, 1)$ order preserving correspondence between the left principal ideals of $\mathfrak{R}$ and the $x \leqslant L(a_j)$; it follows that $e^r{}_j \oplus (1 - e)^r{}_j = a_j$. Now (3.5) follows from (3.4).

Proof of (3.6). Using (1.3),

$$\begin{aligned}(\alpha - \beta)^0{}_k &= [\alpha_{ij} + (a_k + \beta_{ij})(a_j + c_{ik})]a_k \\ &= [\alpha_{ij}(a_k + \beta_{ij}) + (a_k + \beta_{ij})(a_j + c_{ik})]a_k \\ &= [\alpha_{ij}\beta_{ij} + (a_j + c_{ik})(a_k + \beta_{ij})]a_k \\ &= (\alpha_{ij}\beta_{ij} + a_j + c_{ik})(a_k + \beta_{ij})a_k \\ &= (\alpha_{ij}\beta_{ij} + a_j + c_{ik})a_k.\end{aligned}$$

Hence

$$\begin{aligned}(\alpha - \beta)^0{}_i &= [(\alpha - \beta)^0{}_k + c_{ik}]a_i \\ &= (\alpha_{ij}\beta_{ij} + a_j + c_{ik})(a_k + c_{ik})a_i \\ &= (\alpha_{ij}\beta_{ij} + a_j)a_i.\end{aligned}$$

Proof of (3.7). Using (1.3),

$$\begin{aligned}(\alpha - \beta)_{kj} + \beta^r{}_j &= [\alpha_{ij} + (a_k + \beta_{ij})(a_j + c_{ik})](a_k + a_j) + (\beta_{ij} + a_i)a_j \\ &= [\alpha_{ij} + (\beta_{ij} + a_i)a_j + (a_k + \beta_{ij})(a_j + c_{ik})](a_k + a_j) \\ &= [\alpha_{ij} + (\beta_{ij} + a_i + a_k)a_j + (a_k + \beta_{ij})(a_j + c_{ik})](a_k + a_j) \\ &= [\alpha_{ij} + (\beta_{ij} + a_i + a_k)(a_j + a_k + \beta_{ij})(a_j + c_{ik})](a_k + a_j) \\ &\geqslant (\alpha_{ij} + c_{ik})(a_k + a_j) = \alpha_{kj}.\end{aligned}$$

4. The i-elements in L. We shall call x an *i-element* if $x \leqslant A^i$ and $xA^{i-1} = 0$. We shall prove that for arbitrary L-numbers $\beta^j (j = 1, \ldots, i - 1)$ and arbitrary idempotent e, the formula:

$$x = (e^r{}_i + A^{i-1}) \prod_{j=1}^{i-1} (\beta^j{}_{ij} + A^{i-1}{}_j) \tag{4.1}$$

defines an i-element, (in particular, if $e = 1$, $e^r{}_i + A^{i-1} = a_i + A^{i-1}$ and may be omitted). For clearly $x \leqslant A^i$ and

$$\begin{aligned}xA^{i-1} &= A^{i-1} \prod_{j=1}^{i-1} (\beta^j{}_{ij} + A^{i-1}{}_j) \\ &= A^{i-1} \prod_{j=1}^{i-1} (\beta^j{}_{ij}a_j + A^{i-1}{}_j) \\ &= \prod_{j=1}^{i-1} A^{i-1}{}_j = 0.\end{aligned}$$

For each i-element x we define:

$$x^j = (x + A^{i-1}{}_j)(a_i + a_j), \qquad j = 1, \ldots, i - 1. \tag{4.2}$$

For each choice of idempotent $e = e(x)$ with $e^r{}_i = (x + A^{i-1})a_i$ (such an e exists by (3.1)), we define:

$$B = B(x) = x + e^0{}_i, \quad B^j = (B + A^{i-1}{}_j)(a_i + a_j). \tag{4.3}$$

(B may not be uniquely determined by x.) The following relations hold:

(4.4) $e^r{}_i + A^{i-1} = x + A^{i-1}$, $e^r{}_i + a_j = x^j + a_j$, $x^j + A^{i-1}{}_j = x + A^{i-1}{}_j$,

(4.5) $BA^{i-1} = [x + e^0{}_i(x + A^{i-1})]A^{i-1} = (x + e^0{}_i e^r{}_i)A^{i-1} = 0$, by (3.5),
$B + A^{i-1} = x + A^{i-1} + e^0{}_i = x + A^{i-1} + e^r{}_i + e^0{}_i$
$= x + A^{i-1} + a_i = A^i$.

(4.6) $B^j a_j = (B + A^{i-1}{}_j)a_j = (BA^{i-1} + A^{i-1}{}_j)a_j = A^{i-1}{}_j a_j = 0$,
$B^j + a_j = (B + A^{i-1}{}_j + a_j)(a_i + a_j) = (B + A^{i-1})(a_i + a_j)$
$= A^i(a_i + a_j) = a_i + a_j$.

(4.7) $B(e^r{}_i + A^{i-1}) = B(x + A^{i-1}) = x + BA^{i-1} = x$,
$B^j(e^r{}_i + a_j) = (B + A^{i-1}{}_j)(x^j + a_j) = x^j + (B + A^{i-1}{}_j)a_j = x^j$.

(4.8) $B^j a_i = (x + e^0{}_i + A^{i-1}{}_j)a_i = e^0{}_i + (x + A^{i-1}{}_j)a_i$
$= e^0{}_i + (x^j + A^{i-1}{}_j)a_i$.

Thus B^j is in L_{ij} for each $j = 1, \ldots, i-1$. Let α^j be the L-number with $\alpha^j{}_{ij} = B^j$. Then

(4.9) $$\alpha^j{}_{ij} \geqslant x^j, \quad e\alpha^j = \alpha^j = e\beta^j$$

for every L-number β^j with $\beta^j{}_{ij} \geqslant x^j$. To verify (4.9): from (4.7), $\alpha^j{}_{ij} = B^j \geqslant x^j$; from (4.8), $(a^j)^0{}_i = B^j a_i \geqslant e^0{}_i = (1 - e)^r{}_i$ and (3.3) then implies $(1-e)\alpha^j = 0$, i.e., $e\alpha^j = \alpha^j$; finally, from (3.6),

$$(\alpha^j - \beta^j)^0{}_i = (B^j\beta^j{}_{ij} + a_j)a_i \geqslant (B^j x^j + a_j)a_i = (e^r{}_i + a_j)a_i = e^r{}_i$$

and by (3.3), $e(\alpha^j - \beta^j) = 0$, that is, $e\alpha^j = \alpha^j = e\beta^j$.

Suppose $\bar{e}$ is also a possible choice of $e(x)$ and let $\bar{\alpha}^j (j = 1, \ldots, i-1)$ denote the corresponding L-numbers. The definition of $e(x)$ shows that $\bar{e}^r = e^r$ and then (3.2) implies $\bar{e}e = \bar{e}$, $e\bar{e} = e$. Now (4.9) implies

(4.9)′ $$\bar{e}\alpha^j = \bar{\alpha}^j, \qquad 1 \leqslant j < i.$$

The following relations follow easily from (4.3) and (4.7):

(4.10) $$B = \prod_{j=1}^{i-1} (\alpha^j{}_{ij} + A^{i-1}{}_j);$$

(4.11) $$x = (e^r{}_i + A^{i-1}) \prod_{j=1}^{i-1} (\alpha^j{}_{ij} + A^{i-1}{}_j).$$

We shall prove that for arbitrary β^j with $\beta^j{}_{ij} \geqslant x^j$ for $j = 1, \ldots, i-1$, the i-element:

(4.12) $$(e^r{}_i + A^{i-1}) \prod_{j=1}^{i-1} (\beta^j{}_{ij} + A^{i-1}{}_j)$$

is identical with x. To prove this let (4.12) be denoted by y. Then

$$\begin{aligned}(e^r{}_i + A^{i-1})(\alpha^j{}_{ij} + A^{i-1}{}_j) &= A^{i-1}{}_j + B^j(e^r{}_i + A^{i-1})(a_i + a_j) \\ &= A^{i-1}{}_j + B^j(e^r{}_i + a_j) = A^{i-1}{}_j + x^j \\ &< A^{i-1}{}_j + \beta^j{}_{ij}\end{aligned}$$

for each $j = 1, \ldots, i-1$, and hence

$$x = (e'_i + A^{i-1}) \prod_{j=1}^{i-1} (\alpha^j{}_{ij} + A^{i-1}{}_j)$$
$$\leqslant (e'_i + A^{i-1}) \prod_{j=1}^{i-1} (\beta^j{}_{ij} + A^{i-1}{}_j) = y,$$

that is, $x \leqslant y$. But since x and y are both i-elements, it follows from the indivisibility of inverses that $x = y$. (For $y + A^{i-1} = x + A^{i-1}$, and hence $y = y(x + A^{i-1}) = x + yA^{i-1} = x$.)

Conversely, for arbitrary idempotent e and arbitrary L-numbers β^j, the formula (4.12) defines an element x in L with the properties: x is an i-element, $x^j \leqslant \beta^j{}_{ij}$ for $j = 1, \ldots, i-1$, and e is a possible choice for $e(x)$. For, $(x + A^{i-1})a_i = e'_i$ and

$$\begin{aligned} x^j &= (x + A^{i-1}{}_j)(a_i + a_j) \\ &= \left[(e'_i + A^{i-1}) \prod_{k=1}^{i-1} (\beta^k{}_{ik} + A^{i-1}{}_k) + A^{i-1}{}_j \right](a_i + a_j) \\ &= (e'_i + A^{i-1})(\beta^j{}_{ij} + A^{i-1}{}_j)(a_i + a_j) \\ &= [(e'_i + a_j)\beta^j{}_{ij} + A^{i-1}{}_j](a_i + a_j) = (e'_i + a_j)\beta^j{}_{ij} \\ &\leqslant \beta^j{}_{ij}, \end{aligned}$$

as stated. Moreover, (4.9) shows that $\alpha^j(x) = e\beta^j$.

5. An important identity. Suppose that $1 \leqslant j < n$, $i = j + 1$, and let δ^m and θ^m, $m = 1, \ldots, j$, and β be arbitrary elements in $\mathfrak{R}$. Then the following identity holds:[4]

$$\begin{aligned} (5.1) \qquad &\prod_{m=1}^{i-1} [(\delta^m + \beta\theta^m)_{im} + A^{i-1}{}_m] \\ &= [(\beta_{ij} + A^{i-1}) \prod_{m=1}^{i-1} (\delta^m{}_{im} + A^j{}_m) + \prod_{m=1}^{i-1} (\theta^m{}_{jm} + A^{i-1}{}_m)](A^{i-1} + a_i). \end{aligned}$$

To prove this identity we shall first establish that for arbitrary indices i, j, m all different, and arbitrary δ, β, θ in $\mathfrak{R}$,

$$(5.2) \qquad (\delta + \beta\theta)_{im} = [\theta_{jm} + (\delta_{im} + a_j)(\beta_{ij} + a_m)](a_i + a_m).$$

Indeed the addition formula (2.1) shows that if k is different from i, j, m, (and there is such a k since we assume $n \geqslant 4$),

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[4] This identity makes possible a simple proof that the module which we shall assign to an x in L (see section 6), is uniquely determined by x (Theorem 6.5). This is a critical step in the proof of the coordinatization theorem as given in the present paper. It will be noted that the proof of this identity uses the same technique that was used by one of us in 1937 to obtain a direct proof of the associativity and distributivity laws in $\mathfrak{R}$ (1, §§ 4.11, 4.12).

left side of (5.2) $= (\beta\theta + \delta)_{tm}$

$= [(\beta\theta)_{km} + (\delta_{tm} + a_k)(c_{tk} + a_m)](a_t + a_m)$

$= [(\beta_{kj} + \theta_{jm})(a_k + a_m + a_t) + (\delta_{tm} + a_k)(c_{tk} + a_m)](a_t + a_m)$ (ST)[6]

$= [\beta_{kj} + \theta_{jm} + (\delta_{tm} + a_k)(c_{tk} + a_m)](a_k + a_m + a_t)(a_t + a_m)$ (ML)

$= [\theta_{jm} + \beta_{kj} + (\delta_{tm} + a_k + a_j)(c_{tk} + a_m)](a_t + a_m)$ (ST)

$= [\theta_{jm} + (\delta_{tm} + a_k + a_j)(\beta_{kj} + c_{tk} + a_m)](a_t + a_m)$ (AL)

$= [\theta_{jm} + (\delta_{tm} + a_j)(\beta_{kj} + c_{tk} + a_m)](a_t + a_m)$ (CI)

$= [\theta_{jm} + (\delta_{tm} + a_j)\{(\beta_{kj} + c_{tk})(a_m + a_t + a_j) + a_m\}](a_t + a_m)$ (CI)

$= [\theta_{jm} + (\delta_{tm} + a_j)\{(\beta_{kj} + c_{tk})(a_t + a_j) + a_m\}](a_t + a_m)$

$=$ (right side of (5.2)).

This proves (5.2). It follows that

$$\text{(left side of (5.1))} = \prod_{m=1}^{j-1} [\{\theta^m{}_{jm} + (\delta^m{}_{tm} + a_j)(\beta_{tj} + a_m)\}(a_t + a_m) + A^{j-1}{}_m]$$

$$= (a_t + A^{j-1}) \prod [(\theta^m{}_{jm} + A^{j-1}{}_m) + (\delta^m{}_{tm} + A^j{}_m)(\beta_{tj} + A^{j-1})]$$

$$> \text{(right side of (5.1))}.$$

But both the left side and the right side of (5.1) are inverses of A^{j-1} in $a_t + A^{j-1}$; for the left side of (5.1) is a j-element and hence:

A^{j-1} (right side of (5.1)) $< A^{j-1}$ (left side of (5.1)) $= 0$, and

$$a_t + A^{j-1} > \text{(left side of (5.1))} + A^{j-1}$$

$$> \text{(right side of (5.1))} + A^{j-1}$$

$$= (a_t + A^{j-1})[\prod_{m=1}^{j-1} (\theta^m{}_{jm} + A^{j-1}{}_m) + (\beta_{tj} + A^{j-1}) \prod_{m=1}^{j-1} (\delta^m{}_{tm} + A^j{}_m) + A^{j-1}]$$

$$= (a_t + A^{j-1})[\prod (\theta^m{}_{jm} + A^{j-1}{}_m) + (\beta_{tj} + A^{j-1}) \prod (\delta^m{}_{tm} + A^j)]$$

$$= (a_t + A^{j-1})[\prod (\theta^m{}_{jm} + A^{j-1}{}_m) + (\beta_{tj} + A^{j-1}) \prod (a_t + A^j)]$$

$$= (a_t + A^{j-1})[\prod (\theta^m{}_{jm} + A^{j-1}{}_m) + \beta_{tj} + A^{j-1}]$$

$$= (a_t + A^{j-1})[\prod (\theta^m{}_{jm} + A^{j-1}{}_m + a_m) + \beta_{tj}]$$

$$= (a_t + A^{j-1})[\prod (A^j) + \beta_{tj}]$$

$$= (a_t + A^{j-1})(a_t + A^j)$$

$$= (a_t + A^{j-1}).$$

Now the indivisibility of inverses shows that equality holds in (5.1).

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[6](ST) will indicate application of the superfluous term identity: $ab = a(b + c) = a\,b\,c$ for all a, b, c, d with $c(a + b) = 0$ and $b \leqslant d$; similarly, for application of the modular law we use (ML); for the absorption law: $ab + c = a(b + c)$ for all a, b, c with $c \leqslant a$ we use (AL); and for the clipping identity $a(b + c) = a[b(a + c) + c]$ for all a, b, c we use (CI) (1, Section 2.2).

As a corollary to (5.1) we shall derive the following identity which holds for arbitrary β^m and θ^m, $m = 1, \ldots, j$, $i = j + 1$, and arbitrary γ:

$$(5.3)\quad [(\beta^j - \gamma)_{ij} + A^{i-1}]\prod_{m=1}^{j-1}[(\beta^m + \gamma\theta^m)_{im} + A^j{}_m] + \prod_{m=1}^{j-1}(\theta^m{}_{jm} + A^{i-1}{}_m)$$
$$= (\beta^j{}_{ij} + A^{i-1})\prod_{m=1}^{j-1}(\beta^m{}_{im} + A^j{}_m) + \prod_{m=1}^{j-1}(\theta^m{}_{jm} + A^{i-1}{}_m).$$

The right side of (5.3) is precisely the left side of (5.3) with a particular value for γ, $\gamma = 0$. Thus we need only show that for any γ the left side of (5.3) has the same value, equal to its value when $\gamma = \beta^j$, say. For this purpose it is sufficient to establish the identity:

$$(5.4)\quad (\text{left side of } (5.3)) = \prod[(\beta^m + \beta^j\theta^m)_{im} + A^{i-1}{}_m] + \prod(\theta^m{}_{jm} + A^{i-1}{}_m).$$

Now (5.4) can be obtained by substituting in (5.1): $\beta^j - \gamma$ for β and $\beta^m + \gamma\theta^m$ for δ^m, $m = 1, \ldots, j-1$ and adding the term $\Pi(\theta^m{}_{jm} + A^{i-1}{}_m)$ to both sides.

6. The rule for assigning left modules to elements of L. For each x in L call $x_1, \ldots, x_n$ a *base-decomposition* of x if each x_i is an inverse:

$$x_i = [xA^i - xA^{i-1}], \qquad i = 1, \ldots, n.$$

Clearly each x_i is an i-element and $x = x_1 \oplus \ldots \oplus x_n$. For each base-decomposition of x and for any idempotent e^i satisfying:

$$(e^i)^r{}_i = (x_i + A^{i-1})a_i = (xA^i + A^{i-1})a_i,$$

let $B(x_i)$, $B^j(x_i)$, $\alpha^{ij} = \alpha^j(x_i)$ be determined as in §4, and define the vector

$$u(x_i) = (-\alpha^{i1}, \ldots, -\alpha^{i,i-1}, e^i, 0, \ldots, 0).$$

Now, for each such $u(x_i)$, $i = i, \ldots, n$, assign to x the left module

$$M(x_1, \ldots, x_n) = (u(x_1), \ldots, u(x_n))_l.$$

We note that: (i) the x_i may not be uniquely determined by x and for each x_i the idempotent e^i may not be uniquely determined by x_i, however it follows from (4.9)′ that $(u(x_i))_l$ is uniquely determined by x_i so that $M(x_1, \ldots, x_n)$ is uniquely determined by $x_1, \ldots, x_n$; (ii) if x is a j-element then the x_i *are* uniquely determined with $x_i = x$ for $i = j$ and $x_i = 0$ for $i \neq j$; (iii) if x_i is an arbitrary i-element for each $i = 1, \ldots, n$, and $x = x_1 + \ldots + x_n$ then $x_1, \ldots, x_n$ is a base-decomposition for x.

We shall prove below the following statements (6.1)–(6.7).

(6.1) Every left module M of finite span is identical with $M(x_1, \ldots, x_n)$ for some base-decomposition $x_1, \ldots, x_n$ of some x in L.

(6.2) Suppose $x_1, \ldots, x_n$ and $\bar{x}_1, \ldots, \bar{x}_n$ are base-decompositions for the same x. If

$$u(x_m) = (-\alpha^{m,1}, \ldots, -\alpha^{m,m-1}, e^m, 0, \ldots, 0)$$

and

$$u(\bar{x}_m) = (-\bar{\alpha}^{m,1}, \ldots, -\bar{\alpha}^{m,m-1}, \bar{e}^m, 0, \ldots, 0)$$

and the $\alpha^{m,k}$ and the $\bar{\alpha}^{m,k}$ both form canonical matrices[6] with $\bar{e}^m = e^m$, then $M(x_1, \ldots, x_n) = M(\bar{x}_1, \ldots, \bar{x}_n)$.

(6.3). Suppose y is an i-element and z is a j-element with $1 \leqslant j < i \leqslant n$. If for some γ in $\mathfrak{R}$ there is a relation $u(x) = u(y) + \gamma u(z)$ for some i-element x, then $x \leqslant y + z$; on the other hand if e is the ith coordinate of $u(y)$ and γ satisfies $e\gamma = \gamma$ then the vector $u(y) + \gamma u(z)$ has ith coordinate $= e$ and coincides with $u(x)$ for some i-element x with $z + x = z + y$.

(6.4). For each base-decomposition $x_1, \ldots, x_n$ of x and choice of the $u(x_m)$ there exists a base-decomposition $\bar{x}_1, \ldots, \bar{x}_n$ of x and choices of $u(\bar{x}_m)$:

$$u(\bar{x}_m) = (-\bar{\alpha}^{m,1}, \ldots, -\bar{\alpha}^{m,m-1}, e^m, 0, \ldots, 0)$$

such that

(6.4)′ $M(\bar{x}_1, \ldots, \bar{x}_n) = M(x_1, \ldots, x_n)$,

(6.4)″ $\bar{\alpha}^{m,k} e^k = 0$ for all $n \geqslant m > k \geqslant 1$,

implying that the $\bar{\alpha}^{m,k}$ form a canonical matrix.

(6.5) For each x in L, all $M(x_1, \ldots, x_n)$ assigned to x coincide, so that we may write $M(x)$ for $M(x_1, \ldots, x_n)$.

(6.6) $x \leqslant y$ implies that $M(x) \leqslant M(y)$.

(6.7) $M(x) \leqslant M(y)$ implies $x \leqslant y$.

The coordinatization theorem follows easily from (6.1), (6.5), (6.6) and (6.7).

7. Proof of (6.1). M is spanned by some canonical basis $u^i = (\alpha^{i1}, \ldots, \alpha^{in})$, $i = 1, \ldots, n$ (**1**, §3.4). Choose e^i, $x(i)$ as follows:

$$e^i = \alpha^{ii},$$

$$x(i) = [(e^i)'_i + A^{i-1}] \prod_{j=1}^{i-1} [(-\alpha^{ij})_{ij} + A^{i-1}{}_j].$$

[6] It has been shown (**5**, Theorem 2.12 and Lemma 2.11; **1**, Section 3.4) that a left module of finite span is always spanned by n vectors $(\alpha^{j1}, \ldots, \alpha^{jn})$, $j = 1, \ldots, n$, with the properties: for each j, α^{jj} is idempotent, $= e^j$, say; for all $i > j$, $\alpha^{ji} = 0$; for all $i < j$, $e^j \alpha^{ji} = \alpha^{ji}$ and $\alpha^{ji} e^i = 0$. Such a set of n vectors is called a *canonical basis* for the left module. A matrix whose rows form such a canonical basis is called a *canonical matrix*.

Then § 4 shows that each $x(i)$ is an i-element and that e^i is a possible choice for $e(x(i))$; with this choice of $e(x(i))$ it follows from the last paragraph of §4 that $\alpha^j(x(i)) = -\alpha^{ij}$, (for a canonical basis $e^i\alpha^{ij} = \alpha^{ij}$), that u^i is a possible choice for $u(x(i))$ and hence M coincides with $M(x_1, \ldots, x_n)$.

8. Proof of (6.2). We shall show that $\bar{x}_m = x_m$ for all m so that the α^{mk} are uniquely determined by x and the e^m (if the α^{mk} are to form a canonical matrix).

Set $U^k = (e^k)^0{}_k \leqslant a_k$ for each $k < m$. Since (3.5) shows that $(e^k)^0{}_k\,(e^k)^r{}_k = 0$, it follows that

$$U^k(x_k + A^{k-1}) = 0. \tag{8.1}$$

We shall show that

$$x_m = x\prod_{k=1}^{m-1}(U^k + A^m{}_k); \tag{8.2}$$

this will establish the uniqueness of x_m since the U^k are uniquely determined by the e^k.

From §4 there is a $B(x_m) \geqslant x_m$ for which

$$\alpha^{mk}{}_{mk} = (B(x_m) + A^{m-1}{}_k)(a_m + a_k);$$

now (3.3) implies that $(\alpha^{mk})^r{}_k \leqslant (e^k)^0{}_k$; i.e., $(B(x_m) + A^m{}_k)a_k \leqslant U^k$. Hence $U^k + A^m{}_k \geqslant B(x_m) \geqslant x_m$ for each k and so (right side of (8.2)) $\geqslant x_m$. Now

$$\text{(right side of (8.2))} = xA^m\prod_{k=1}^{m-1}(U^k + A^m{}_k) = (x_1 + \ldots + x_m)\prod_{k=1}^{m-1}(U^k + A^m{}_k)$$

$$= x_m + (x_1 + \ldots + x_{m-1})\prod_{k=1}^{m-1}(U^k + A^m{}_k). \tag{8.3}$$

But

$$(x_1 + \ldots + x_{m-1})\prod_{k=1}^{m-1}(U^k + A^m{}_k) = (x_1 + \ldots + x_{m-1})\prod_{k=1}^{m-1}(U^k + A^{m-1}{}_k) \tag{8.4}$$

$$= (x_1 + \ldots + x_{m-1})(U^{m-1} + A^{m-2})\prod_{k=1}^{m-2}(U^k + A^{m-1}{}_k)$$

$$= (x_1 + \ldots + x_{m-2})\prod_{k=1}^{m-2}(U^k + A^{m-1}{}_k)$$

since (8.1) and (CI) show that $x_{m-1}(U^{m-1} + A^{m-2}) = 0$. Repetition of the reduction in (8.4) shows that the second addend of (8.3) is zero and establishes (8.2).

9. Proof of (6.3). If e is the ith coordinate of $u(y)$ with y an i-element and z a j-element, $1 \leqslant j < i \leqslant n$, and γ is an element of $\mathfrak{R}$ with $e\gamma = \gamma$, then $u(y) + \gamma u(z)$ is a vector of the form:

$$(-e\beta^1, -e\beta^2, \ldots, -e\beta^{i-1}, e, 0, \ldots, 0),$$

and hence coincides with $u(x)$ for some i-element x, by the last paragraph of §4. This proves the second part of (6.3) except for the relation $z + x = z + y$.

However this relation follows easily from the first part of (6.3), which we now prove.

Because of §4 the first part of (6.3) is equivalent to the statement:

$$(9.1)\qquad (\bar{e}^r{}_i + A^{i-1}) \prod_{m=1}^{i-1} (\alpha^m{}_{im} + A^{i-1}{}_m)$$
$$\leqslant (\bar{e}^r{}_i + A^{i-1}) \prod_{m=1}^{i-1} (\beta^m{}_{im} + A^{i-1}{}_m) + (e^r{}_j + A^{j-1}) \prod_{m=1}^{j-1} (\theta^m{}_{jm} + A^{j-1}{}_m)$$

for arbitrary idempotents $\bar{e}$ and e in $\mathfrak{R}$ provided that $\bar{e}\alpha^m = \alpha^m$ for $m < i$, $\bar{e}\beta^m = \beta^m$ for $m < i$, $e\theta^m = \theta^m$ for $m < j$ and

$$(9.1)'\qquad \begin{cases} \alpha^m = \beta^m & \text{for } j < m \leqslant i, \\ \alpha^j = \beta^j - \gamma e & \text{for some } \gamma, \\ \alpha^m = \beta^m + \gamma\theta^m & \text{for } 1 \leqslant m < j. \end{cases}$$

Now (9.1) can be deduced from the (apparently) simpler statement:

$$(9.2)\qquad \prod_{m=1}^{i-1} (\alpha^m{}_{im} + A^{i-1}{}_m) \leqslant \prod_{m=1}^{i-1} (\beta^m{}_{im} + A^{i-1}{}_m) + (e^r{}_j + A^{j-1}) \prod_{m=1}^{j-1} (\theta^m{}_{jm} + A^{j-1}{}_m)$$

for arbitrary idempotent e and arbitrary θ^m satisfying $e\theta^m = \theta^m$ for $m < j$ and arbitrary α^m, β^m satisfying (9.1)′ for some γ. Indeed, clipping (9.2) by $\bar{e}^r{}_i + A^{i-1}$ yields (9.1) as required.

If $j < i - 1$ then $\alpha^m = \beta^m$ for $j + 1 \leqslant m \leqslant i - 1$ and (9.2) can be deduced from:

$$(9.3)\qquad \prod_{m=1}^{j} (\alpha^m{}_{im} + A^{i-1}{}_m) \leqslant \prod_{m=1}^{j} (\beta^m{}_{im} + A^{i-1}{}_m) + (e^r{}_j + A^{j-1}) \prod_{m=1}^{j-1} (\theta^m{}_{jm} + A^{j-1}{}_m);$$

for clipping (9.3) by

$$\prod_{m=j+1}^{i-1} (\alpha^m{}_{im} + A^{i-1}{}_m)$$

gives (9.2).

Since $a_1, \ldots, a_i$ are independent, we may write (9.3) (use **(1, §2.3)**) as:

$$(9.4)\qquad (a_{j+1} + \ldots + a_{i-1}) + \prod_{m=1}^{j} (\alpha^m{}_{im} + A^j{}_m)$$
$$\leqslant (a_{j+1} + \ldots + a_{i-1}) + \prod_{m=1}^{j} (\beta^m{}_{im} + A^j{}_m) + (e^r{}_j + A^{j-1}) \prod_{m=1}^{j-1} (\theta^m{}_{jm} + A^{j-1}{}_m)$$

and this would clearly be a consequence of

$$(9.5)\qquad \prod_{m=1}^{j} (\alpha^m{}_{im} + A^j{}_m) \leqslant \prod_{m=1}^{j} (\beta^m{}_{im} + A^j{}_m) + (e^r{}_j + A^{j-1}) \prod_{m=1}^{j-1} (\theta^m{}_{jm} + A^{j-1}{}_m).$$

But (9.5) can be derived from

$$(9.6)\qquad \prod_{m=1}^{j} (\alpha^m{}_{im} + A^j{}_m) \leqslant \prod_{m=1}^{j} (\beta^m{}_{im} + A^j{}_m) + \prod_{m=1}^{j-1} (\theta^m{}_{jm} + A^{j-1}{}_m)$$

by clipping both sides of (9.6) by $\beta^j{}_{ij} + e^r{}_j + A^{j-1}$: indeed, this clipping does not change the left side of (9.6) since one of its factors is

$$\begin{aligned} \alpha^j{}_{ij} + A^{j-1} &= (\beta^j - \gamma e)_{ij} + A^{j-1} \\ &\leqslant \beta^j{}_{ij} + (-\gamma e)^r{}_j + A^{j-1} && \text{using (3.7)} \\ &\leqslant \beta^j{}_{ij} + e^r{}_j + A^{j-1} && \text{using (3.2);} \end{aligned}$$

on the other hand this clipping changes the right side of (9.6) to

$$\prod_{m=1}^{j} (\beta^m{}_{im} + A^j{}_m) + (\beta^j{}_{ij} + e^r{}_j + A^{j-1}) \prod_{m=1}^{j-1} (\theta^m{}_{jm} + A^{j-1}{}_m) \qquad \text{(ML)}$$

$$= \prod_{m=1}^{j} (\beta^m{}_{im} + A^j{}_m) + (e^r{}_j + A^{j-1}) \prod_{m=1}^{j-1} (\theta^m{}_{jm} + A^{j-1}{}_m)$$

since the modular law implies $(\beta^j{}_{ij} + e^r{}_j + A^{j-1})A^j = e^r{}_j + A^{j-1}$.

But since $\alpha^m = \beta^m + \gamma\theta^m$ for $m < j$ and $\alpha^j = \beta^j - \gamma$, the desired (9.6) follows immediately from (5.3) proved above.

10. Proof of (6.4). By (6.3) there exist $\bar{x}_2, \ldots, \bar{x}_n$ with $x_1 + \bar{x}_m = x_1 + x_m$ for all m (so that $x_1, \bar{x}_2, \ldots, \bar{x}_n$ is again a base-decomposition of x) and with $u(\bar{x}_m) = u(x_m) + \alpha^{m,1}\, u(x_1)$ so that $\bar{\alpha}^{m,1} e^1 = 0$ for $m > 1$. Similarly $\bar{x}_3, \ldots, \bar{x}_n$ can be replaced so that the new $\bar{\alpha}^{i,j}$ satisfy also $\bar{\alpha}^{m,2} e^2 = 0$ for $m > 2$. Successive repetition of this procedure establishes (6.4).

11. Proof of (6.5). If $M = M(x_1, \ldots, x_n) = (u(x_1), \ldots, u(x_n))_l$ we may, without changing M, replace $u(x_m)$ by $e^m\, u(x_m)$ where e^m is *any* idempotent satisfying $(e^m)^r{}_m = (xA^m + A^{m-1})a_m$. The statement (6.5) now follows from (6.4) and (6.2).

12. Proof of (6.6). If $x \leqslant y$ then we may choose the x_i, y_i so that $x_i \leqslant y_i$ (for example, choose $y_i = x_i + [yA^i - (yA^{i-1} + x_i)]$. Then $(e(x_i))^r{}_i \leqslant (e(y_i))^r{}_i$ which implies $e(x_i)\, e(y_i) = e(x_i)$; we may choose $\beta^j(x_i)$ to coincide with $\beta^j(y_i)$ since

$$\beta^j{}_{ij}(y_i) \geqslant (y_i + A^{i-1}{}_j)(a_i + a_j)$$

which implies $\beta^j{}_{ij}(y_i) \geqslant (x_i + A^{i-1}{}_j)(a_i + a_j)$. Now $e(x_i)u(y_i) = u(x_i)$ for each $i = 1, \ldots, n$ and hence $M(x_1, \ldots, x_n) \leqslant M(y_1, \ldots, y_n)$. Because of (6.5) it follows that $M(x) \leqslant M(y)$.

13. Proof of (6.7). Since x is a union of i-elements, $i = 1, \ldots, n$, it is sufficient to prove (6.7) with the restriction that x is an i-element. Then if $y_1, \ldots, y_n$ is a base-decomposition of y, $u(x)$ will have a representation

$$u(x) = \gamma_1\, u(y_1) + \ldots + \gamma_n\, u(y_n).$$

Let $e(x)$ be the ith coordinate of $u(x)$ and $e(y_m)$ the mth coordinate of $u(y_m)$. Then $e(x)\, u(x) = u(x)$ and replacing γ_m by $e(x)\gamma_m$ we may suppose $e(x)\gamma_m = \gamma_m$ for all m.

Now if x is an i-element (i fixed) and $i < n$ then the nth coordinates of $u(x)$ and of $\gamma_m\, u(y_m)$, for $m < n$, are all 0 and hence the nth coordinate $\gamma_n\, e(y_n) = 0$. This implies $\gamma_n\, u(y_n)$ is the zero vector. Successive applications of this argument show that $\gamma_m\, u(y_m) = 0$ for all $m > i$ and that $e(x) = \gamma_i\, e(y_i)$.

The last paragraph of §4 shows that $\gamma_i\, u(y_i) = u(\bar{y}_i)$ for some i-element $\bar{y}_i$. Then (4.12) shows that $\bar{y}_i \leqslant y_i$ since $(e(y_i))'_i \geqslant (\gamma_i\, e(y_i))'_i$. Hence we need only prove (6.7) with y_i replaced by $\bar{y}_i$; thus we may suppose that $\bar{y}_i = y_i$, that is, we may suppose that $\gamma_i = 1$, and $e(y_i) = e(x) = e$ (say) and $e\gamma_m = \gamma_m$ for all $m < i$. Now (6.3) shows that $u(y_i) + \gamma_{i-1}\, u(y_{i-1}) = u(z)$ for some i-element z with $z \leqslant y_i + y_{i-1}$. Similarly $u(z) + \gamma_{i-2}\, u(y_{i-2}) = u(\bar{z})$ for some i-element $\bar{z}$ with $\bar{z} \leqslant z + y_{i-2} \leqslant y_i + y_{i-1} + y_{i-2}$. Repetition of this argument finally yields $x \leqslant y_i + y_{i-1} + \ldots + y_1 \leqslant y$ as required.

This completes the proof of all statements (6.1) to (6.7) and establishes the coordinatization theorem.

Added in proof. A book, in Japanese, by F. Maeda, entitled Continuous Geometry (Tokyo, 1952), is reported to contain a simplification of von Neumann's coordinatization theorem, based on methods of K. Kodaira and S. Huruya (see Mathematical Reviews, *15* (1954), p. 540). We regret that we have not been able to see any further details of this work.

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SUBSPACES OF RIEMANNIAN SPACES

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Summary. In this paper, results obtained by the author for Riemannian Spaces V_n imbedded in Euclidean Spaces E_N **(3; 4)** are extended to V_n imbedded in V_N.

The first section is introductory. In §2 the general result is obtained. This is the establishment of a certain dependency among the three basic sets of equations of the V_n with respect to the V_N, namely the equations of Gauss, Codazzi and Kuehne. In §3 it is assumed that V_N is of constant curvature with $N = n + 1$. This case is discussed with the help of a generalization of the type number τ introduced by Thomas **(10)**.

Throughout the paper the conventional tensor notation has been adopted. Capital latin indices vary from 1 to N, small latin indices from 1 to n, and small greek indices from 1 to $\nu = N - n$. Whenever an index occurs twice in an expression, the summation with respect to that index has to be performed, except when otherwise stated. This summation convention is not restricted to indices with opposite (i.e. one of covariant and the other of contravariant) character.

1. Introduction. We consider a V_N given by the positive definite metric:

$$dS^2 = A_{IJ}dX^I dX^J \quad |A_{IJ}| \neq 0 \tag{1}$$

in which the A_{IJ} are continuous functions of the X^I, having continuous partial derivatives up to the third order; and a V_n whose metric

$$ds^2 = a_{ij}dx^i dx^j \quad (n < N) \tag{2}$$

satisfies similar conditions with respect to the x^i.

A set of necessary conditions for the V_n to be imbedded[1] in the V_N is given by the following equations **(8**, no. 47**)**, known respectively as the equations of Gauss, Codazzi and Kuehne:

$$\text{(I)} \quad G_{ijkl} \equiv r_{ijkl} - (b_{\alpha|ik}b_{\alpha|jl} - b_{\alpha|il}b_{\alpha|jk}) - R_{IJKL}X^I_{,i}X^J_{,j}X^K_{,k}X^L_{,l} = 0,$$

$$\text{(II)} \quad C_{\alpha|ijk} \equiv b_{\alpha|ij,k} - b_{\alpha|ik,j} - (t_{\beta\alpha|k}b_{\beta|ij} - t_{\beta\alpha|j}b_{\beta|ik}) + R_{IJKL}\xi^I_\alpha X^J_{,i}X^K_{,j}X^L_{,k} = 0,$$

$$\text{(III)} \quad K_{\alpha\beta|ij} \equiv t_{\alpha\beta|i,j} - t_{\alpha\beta|j,i} + (t_{\gamma\alpha|i}t_{\gamma\beta|j} - t_{\gamma\alpha|j}t_{\gamma\beta|i}) + a^{kl}(b_{\alpha|ki}b_{\beta|lj} - b_{\alpha|kj}b_{\beta|li}) + R_{IJKL}\xi^I_\alpha\xi^J_\beta X^K_{,i}X^L_{,j} = 0.$$

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[1]Throughout this paper, by "imbedding" is meant *local* and *isometrical* imbedding.

Here, r_{ijkl} and R_{IJKL} are the components of the covariant curvature tensor in V_n and V_N respectively. The ξ_α^I are $N-n$ contravariant vectors in V_N of unit length, perpendicular to one another and to the V_n. The $b_{\alpha|ij} = b_{\alpha|ji}$ are coefficients of the fundamental forms of the second kind and $t_{\alpha\beta|i} = -t_{\beta\alpha|i}$ are the "torsions". The index after the comma denotes covariant differentiation with respect to the tensor a_{ij} given by (2).

If V_N is of constant curvature K_0 we have

$$R_{IJKL} = K_0(A_{IK}A_{JL} - A_{IL}A_{JK}),$$

and equations (I), (II), (III) become:

$$(I') \quad G_{ijkl} \equiv r_{ijkl} - (b_{\alpha|il}b_{\alpha|jk} - b_{\alpha|ik}b_{\alpha|jl}) - K_0(a_{il}a_{jk} - a_{ik}a_{jl}) = 0,$$

$$(II') \quad C_{\alpha|ijk} \equiv b_{\alpha|ij,k} - b_{\alpha|ik,j} - (t_{\beta\alpha|k}b_{\beta|ij} - t_{\beta\alpha|j}b_{\beta|ik}) = 0,$$

$$(III') \quad K_{\alpha\beta|ij} \equiv t_{\alpha\beta|i,j} - t_{\alpha\beta|j,i} + (t_{\gamma\alpha|i}t_{\gamma\beta|j} - t_{\gamma\alpha|j}t_{\gamma\beta|i}) + a^{kl}(b_{\alpha|ki}b_{\beta|lj} - b_{\alpha|kj}b_{\beta|li}) = 0.$$

It can be shown that in this case the equations (I'), (II'), (III') are both *necessary and sufficient* conditions for the V_n to be imbedded in the V_N. For a similar problem, see **(10**, pp. 178–182**)**.

As in the case of a V_n in an E_N **(3**; **4)** the question arises at this point whether all the equations (I), (II), (III) are independent. The following section is devoted to answering this question.

2. Independence considerations. The lefthand sides of (I), (II), (III) are obviously tensors in the V_n, which depend in general also upon the V_N. They are denoted by G_{ijkl}; $C_{\alpha|ijk}$; $K_{\alpha\beta|ij}$ and named respectively the tensors of Gauss, Codazzi and Kuehne of the V_n with respect to the V_N **(3**, p. 167f**)**. We can now reformulate the statement in §1 in the following way:

A necessary condition for the V_n to be imbedded in the V_N is that the tensors of Gauss, Codazzi and Kuehne of the V_n with respect to the V_N should vanish.

We are thus able to consider directly the tensors just introduced and certain combinations of their covariant derivatives. This will lead us to discover in certain cases how many of the conditions (I), (II), (III) are independent.

We define **(4**; **6)**:

$$(A) \qquad G_{ijklm} = G_{ijkl,m} + G_{ijlm,k} + G_{ijmk,l},$$

$$(B) \qquad C_{\alpha|ijkl} = C_{\alpha|ijk,l} + C_{\alpha|il,jk} + C_{\alpha|ilj,k},$$

$$(C) \qquad K_{\alpha\beta|ijk} = K_{\alpha\beta|ij,k} + K_{\alpha\beta|jk,i} + K_{\alpha\beta|ki,j}.$$

These tensors will be appropriately named the "derived tensors" of Gauss, Codazzi and Kuehne of the V_n with respect to the V_N. If we perform the indicated calculations, we obtain:

$$(A') \quad G_{ijklm} = - b_{\alpha|ik}C_{\alpha|jlm} - b_{\alpha|ij}C_{\alpha|jmk} - b_{\alpha|im}C_{\alpha|jkl} + b_{\alpha|jk}C_{\alpha|ilm} + b_{\alpha|ji}C_{\alpha|mk} + b_{\alpha|jm}C_{\alpha|ikl},$$

$$(B') \quad C_{\alpha|ijkl} = t_{\beta\alpha|j}C_{\beta|ikl} + t_{\beta\alpha|k}C_{\beta|ijl} + t_{\beta\alpha|l}C_{\beta|ijk} - b_{\beta|ij}K_{\beta\alpha|kl} - b_{\beta|ik}K_{\beta\alpha|jl} - b_{\beta|il}K_{\beta\alpha|jk} + a^{mp}(b_{\alpha|mj}G_{pikl} + b_{\alpha|mk}G_{pijl} + b_{\alpha|ml}G_{pijk}),$$

$$(C') \quad K_{\alpha\beta|ijk} = t_{\gamma\alpha|i}K_{\gamma\beta|jk} + t_{\gamma\alpha|j}K_{\gamma\beta|ki} + t_{\gamma\alpha|k}K_{\gamma\beta|ij} - t_{\gamma\beta|i}K_{\gamma\alpha|jk} - t_{\gamma\beta|j}K_{\gamma\alpha|ki} - t_{\gamma\beta|k}K_{\gamma\alpha|ij} + a^{mp}(b_{\alpha|mi}C_{\beta|pjk} + b_{\alpha|mj}C_{\beta|pki} + b_{\alpha|mk}C_{\beta|pij}) - a^{mp}(b_{\beta|mi}C_{\alpha|pjk} + b_{\beta|mj}C_{\alpha|pki} + b_{\beta|mk}C_{\alpha|pij}).$$

We notice that these derived tensors[2] do not depend explicitly upon the V_N, the last terms from (I), (II), (III) having disappeared. They have therefore the same form as the corresponding derived tensors of the V_n with respect to an E_N **(6**, p. 90**)**. This remarkable fact enables us to extend the results of **(3**; **4**; **6)** to the present case.

These results are essentially based upon the consideration of the tensors (A′), (B′), (C′) and the number of their components. Thus if, for instance, the Gauss tensor G_{ijkl} vanishes in V_n, then the derived Gauss tensor G_{ijklm} is also zero and (A′) becomes a system of linear and homogenous equations in the $C_{\alpha|ijk}$ which reduces, of course, the number of independent components of the Codazzi tensors $C_{\alpha|ijk}$. It is thus possible that, under conditions to be specified below, all the components of the Codazzi tensors $C_{\alpha|ijk}$ vanish *as a result* of the vanishing of Gauss' tensor G_{ijkl}. Similar considerations are valid with respect to (B′) and (C′).

It is necessary at this point to list the number of components of the different tensors introduced so far **(2**; **3**, pp. 170, 174; **6**, p. 91**)**:

$$G_{ijkl} \ldots n^2(n^2-1)/12$$
$$C_{\alpha|ijk} \ldots \nu n(n^2-1)/3$$
$$K_{\alpha\beta|ij} \ldots \nu(\nu-1)\,n(n-1)/4$$
$$G_{ijklm} \ldots n^2(n^2-1)(n-2)/24$$
$$C_{\alpha|ijkl} \ldots \nu n(n^2-1)(n-2)/8$$
$$K_{\alpha\beta|ijk} \ldots \nu(\nu-1)\,n(n-1)(n-2)/12$$

We mention also the result by Burstin **(7)** that, under our assumptions, every V_n can be imbedded in every V_N provided that $N \geqslant \frac{1}{2}n(n+1)$ or $\nu = N - n \geqslant \frac{1}{2}n(n-1)$. It is therefore sufficient to choose $n(n-1)/2$ as the upper limit for ν.

We are now in the position to enunciate the following two theorems:

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[2]The equations obtained by equating (A') and (B') to zero were first used by Allendoerfer **(1)** in the case of a V_n in an E_N to reduce the number of independent equations of (II) and (III).

THEOREM 2.1 *If the equations* (I) *are satisfied by a set of solutions for* $b_{\alpha|ij}$, *for which the ranks of the matrices of the linear systems:*

$$(A'') \quad G_{ijk|lm} \equiv -b_{\alpha|il}C_{\alpha|jkm} - b_{\alpha|ij}C_{\alpha|jml} - b_{\alpha|im}C_{\alpha|jkl} + b_{\alpha|jk}C_{\alpha|ilm} + b_{\alpha|jl}C_{\alpha|iml} + b_{\alpha|jm}C_{\alpha|ikl} = 0,$$

$$(B'') \quad C_{\alpha|ijkl} \equiv -b_{\beta|ij}K_{\beta\alpha|kl} - b_{\beta|ik}K_{\beta\alpha|lj} - b_{\beta|il}K_{\beta\alpha|jk} = 0,$$

have maximum value,[3] *then*

(a) *for* $0 \leqslant \nu \leqslant \frac{1}{3}n(n-2)$, *all equations* (II) *and* (III) *are a consequence of equations* (I);

(b) *for* $\frac{1}{3}n(n-2) \leqslant \nu \leqslant \frac{1}{3}n(n-1)$, *a system of*

$$\tfrac{1}{3}n(n^2-1)[\nu - \tfrac{1}{3}n(n-2)]$$

of equations (II) *are independent. The remainder of equations* (II) *and all the equations* (III) *are a consequence of this system and equations* (I).

THEOREM 2.2. *If the equations* (I) *are satisfied by a set of solutions for* $b_{\alpha|ij}$, *for which the ranks,* r *and* r', *of the matrices of* (A'') *and* (B'') *have not both maximum values, then*

$$\tfrac{1}{3}\nu n(n^2-1) - r$$

of equations (II) *and*

$$\tfrac{1}{4}\nu(\nu-1)\,n(n-1) - r'$$

of equations (III) *remain independent.*

From the table on the previous page it is seen that the matrix of (A'') has $\frac{1}{24}n^2(n^2-1)(n-2)$ rows and $\frac{1}{3}\nu n(n^2-1)$ columns and the matrix of (B'') has $\frac{1}{8}\nu n(n^2-1)(n-2)$ rows and $\frac{1}{4}\nu(\nu-1)\,n(n-1)$ columns. By comparing the two sets of numbers, Theorems 2.1 and 2.2 are readily verified.

In view of this theorem it would seem important to determine the actual ranks of the matrices of (A'') and (B'') in terms of certain numerical invariants of the V_n. Except for the particular case treated in the next section, the author has not succeeded in this task.

In the formation of the tensor $G_{ijk|lm}$ we made use of Bianchi's identities:

$$r_{ijk|l,m} + r_{ijl|m,k} + r_{ijm|k,l} = 0.$$

(Because of this, of course, the number of components of $G_{ijk|lm}$ equals the number of Bianchi's identities **(2)**.)

But Bianchi's identities are a complete set of identities of order one of the tensor of curvature **(5)**. We have therefore the result:

Equations (A'') *are the only ones between the components of* $C_{\alpha|ijk}$, *which can be obtained as a consequence of the validity of equations* (I).

[3] A rectangular matrix with s rows and t columns has maximum rank r if r equals the smaller of the two numbers s, t.

III. It can be seen from equations (I'), (II'), (III') that the problem of imbedding a V_n in a V_N with constant curvature K_0 is equivalent to the problem of imbedding a V_n in a Euclidean E_N, provided that we substitute for the curvature tensor r_{ijkl} of the V_n the tensor:

$$r_{ijkl} - K_0(a_{ik}a_{jl} - a_{il}a_{jk}).$$

We shall also assume $N = n + 1$ in which case (I'), (II'), (III') reduce to:

$$(I'') \quad G_{ijkl} \equiv r_{ijkl} - (b_{ik}b_{jl} - b_{il}b_{jk}) - K_0(a_{ik}a_{jl} - a_{il}a_{jk}) = 0,$$
$$(II'') \quad C_{ijk} \equiv b_{ij,k} - b_{ik,j} = 0,$$

and (A''), (B'') to

$$(A''') \quad G_{ijklm} \equiv - b_{ik}C_{jlm} - b_{il}C_{jmk} - b_{im}C_{jkl} + b_{jk}C_{ilm} + b_{jl}C_{imk} + b_{jm}C_{ikl} = 0.$$

Let τ be the rank of the matrix

$$\|r_{ijkl} - K_0(a_{ik}a_{jl} - a_{il}a_{jk})\|$$

where one of the indices, say i, indicates the rows and the other three indices the columns of the matrix.

It can then be shown that, because of (I''), τ is also the rank of the matrix $\|b_{ij}\|$ **(10**, p. 184**)**.

The integer τ can be considered as an invariant of the V_n with respect to a V_{n+1} of constant curvature K_0 (in the neighbourhood of the point under consideration). It was introduced by Thomas **(10**, loc. cit.**)** for a V_n with respect to an E_{n+1}.

For $\tau = 0$, it follows from (I'') that the V_n is of constant curvature K_0. $\tau = 1$ is impossible. For the remaining values of τ we shall prove

THEOREM 3.1. $\tau > 4$. *All the equations* (II'') *are a consequence of equations* (I'').

THEOREM 3.2. $\tau = 3$. *Of the equations* (II''), *five remain independent. The remainder of the equations* (II'') *are a consequence of these and equations* (I'').

THEOREM 3.3. $\tau = 2$. *Of the equations* (II''), $3n - 4$ *remain independent. The remainder of the equations* (II'') *are a consequence of these and equations* (I'').

Proof. For the values of b_{ij} in the point under consideration we can, by a suitable coordinate transformation, obtain:[4]

$$(3) \quad \begin{aligned} b_{ij} &= 0 && (i \neq j), \\ b_{ii} &\neq 0 && (i = 1, 2, \ldots, \tau), \\ b_{ii} &= 0 && (i = \tau + 1, \ldots, n). \end{aligned}$$

[4]From here to the end of this section, a repeated index does *not* indicate a summation.

For these values, and taking in consideration the basic identities of the tensor G_{ijklm} (which follow readily from its definition):

$$G_{ijklm} = -G_{jiklm}$$
$$G_{ijklm} = G_{ijlmk} = G_{ijmkl} = -G_{ijkml} = -G_{ijmlk} = -G_{ijlkm}$$

the system (A''') reduces to:

$$G_{ijkll} \equiv -b_{ii}C_{jkl} = 0, \tag{4}$$
$$G_{ijkjl} \equiv -b_{ii}C_{jjl} - b_{jj}C_{iil} = 0, \tag{5}$$

where i, j, k, l are all distinct, $i = 1, 2, \ldots \tau$ and $j, k, l = 1, 2, \ldots n$.

Proof of 3.1. $\tau > 4$. From (4) we find then

$$C_{jkl} = 0,$$

and from (5) follows

$$C_{jjl} = 0 \qquad (j = \tau + 1, \tau + 2, \ldots, n).$$

For $j < \tau$, we obtain from (5), by substituting first i for k and then j for k:

$$\begin{aligned} 0 + b_{kk}C_{jjl} + b_{jj}C_{kkl} &= 0, \\ b_{kk}C_{iil} + 0 + b_{ii}C_{kkl} &= 0, \\ b_{jj}C_{iil} + b_{ii}C_{jjl} + 0 &= 0, \end{aligned} \tag{6}$$

for distinct i, j, k, l with $i, j, k = 1, 2, \ldots, \tau$ and $l = 1, 2, \ldots, n$.

The determinant of (6) being different from zero it follows that

$$C_{iil} = 0 \qquad (i \neq l; i = 1, 2, \ldots, \tau; l = 1, 2, \ldots, n).$$

This completes the proof for the case $\tau > 4$ because $C_{iij} \equiv 0$ follows from (II'').

Proof of 3.2. $\tau = 3$. From (4) we obtain

$$C_{jkl} = 0 \qquad (j \neq k \neq l = 1, 2, \ldots, n)$$

provided that at least one of the indices j, k, l, is larger than 3. But the three remaining components of this type, namely C_{123}, C_{231}, and C_{312}, satisfy an identity (which follows easily from (II'')):

$$C_{123} + C_{231} + C_{312} = 0.$$

Thus only two of these components (e.g., C_{123}, C_{231}) remain independent.

From (5) we have

$$C_{jjl} = 0 \qquad (j = 4, 5, \ldots, n; l = 1, 2, \ldots, n).$$

From (6) we obtain for $i = 1, j = 2, k = 3$:

$$\begin{aligned} 0 + b_{33}C_{22l} + b_{22}C_{33l} &= 0, \\ b_{33}C_{11l} + 0 + b_{11}C_{33l} &= 0, \\ b_{22}C_{11l} + b_{11}C_{33l} + 0 &= 0. \end{aligned} \qquad (l = 4, 5, \ldots, n). \tag{7}$$

We have therefore as above:

$$C_{iil} = 0 \qquad (i = 1, 2, 3; l = 4, 5, \ldots, n).$$

From (5) we obtain for $i, j, l = 1, 2, 3$ in turn:

$$b_{23}C_{221} + b_{22}C_{331} = 0,$$

$$b_{23}C_{112} + b_{11}C_{332} = 0,$$

$$b_{22}C_{113} + b_{11}C_{223} = 0.$$

It follows from these equations that three components of the type C_{iil} $(i \neq l = 1, 2, 3)$ are independent (e.g., C_{112}, C_{223}, C_{331}).

Therefore, in the case under consideration *five* components of the tensor C_{ijk} remain independent, namely C_{123}, C_{231}, C_{112}, C_{223}, C_{331}.

Proof of 3.3. $\tau = 2$. From (4) we obtain

$$C_{jkl} = 0, \qquad (j \neq k \neq l = 1, 2, \ldots, n),$$

provided that at least two of the indices are larger than 2. The remaining components of this type are C_{12l}, C_{21l}, C_{11l} among which we have (as before) the identity

$$C_{12l} + C_{21l} + C_{11l} = 0, \qquad (l = 3, 4, \ldots, n).$$

It follows therefore that $2n - 4$ of these components (e.g., C_{12l}, C_{21l}; $l = 3, 4, \ldots, n$) remain independent.

From (5) we have

$$C_{jjl} = 0, \qquad (j = 3, 4, \ldots, n;\ l = 1, 2, \ldots, n),$$

and

$$b_{22}C_{11l} + b_{11}C_{22l} = 0, \qquad (l = 3, 4, \ldots, n).$$

Thus the $n - 2$ components C_{11l} (or C_{22l}) $(l = 3, 4, \ldots, n)$ are independent.

The two components C_{112} and C_{221} are also independent because they do not occur in any of the equations (5).

Therefore, in the case under consideration, $3(n - 2) + 2 = 3n - 4$ components of the tensor C_{ijk} remain independent, namely, C_{112}, C_{221}, C_{12l}, C_{21l}, C_{11l} $(l = 3, 4, \ldots, n)$.

The case $\tau > 4$ of Theorem 3.1 was proved by Thomas **(10, §5)** for a V_n in an E_{n+1}. In its general form but also for a V_n in an E_{n+1}, it was established by the present author **(3, pp. 196ff)**. Here the same line of proof has been adopted.

It is remarkable that in the case $\tau = 3$, the number of independent components of the tensor C_{ijk} does not depend upon the number of dimensions of V_n.

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ON FOURIER-STIELTJES TRANSFORMS

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Let $\mathscr{M}$ be the class of bounded non-decreasing functions defined on the real line which are normalized by the conditions $\phi(-\infty) = 0$, $\phi(t+0) = \phi(t)$. Let $\mathscr{F}$ be the class of Fourier-Stieltjes transforms of elements of $\mathscr{M}$, i.e. the elements of $\mathscr{M}$ and $\mathscr{F}$ are connected by the relation[1]

$$\Phi(x) = \int e^{itx} d\phi(t),$$

where $\phi \in \mathscr{M}$ and $\Phi \in \mathscr{F}$. It is well known, and easy to verify that this mapping from $\mathscr{M}$ to $\mathscr{F}$ is one to one (**1**, p. 67, Satz 18).

It is the purpose of this paper to give various topologies to $\mathscr{F}$ and $\mathscr{M}$ so that the mapping from $\mathscr{F}$ to $\mathscr{M}$ will be continuous or at least continuous at certain points of $\mathscr{F}$ depending on the topologies. The topologies which we shall have occasion to use are enumerated below.

A. The almost weak topology on $\mathscr{F}$. As a neighbourhood basis of an element $\Phi_0 \in \mathscr{F}$ we shall take the sets in $\mathscr{F}$ which satisfy the relations

$$\left| \int f_k(x)[\Phi(x) - \Phi_0(x)]\, dx \right| < \delta, \qquad k = 1, 2, \ldots, n,$$

and

$$\Phi(0) < \Phi_0(0) + \delta$$

where $\{f_k\}_1^n$ is any finite set of elements in the Lebesgue class $L^1(-\infty, \infty)$, and δ is any positive number. We shall designate such neighborhoods by

$$\mathfrak{M}[\{f_k\}; \delta; \Phi_0].$$

B. The mean value topology in $\mathscr{F}$. As a neighborhood basis of an element $\Phi_0 \in \mathscr{F}$ we shall take the sets in $\mathscr{F}$ which satisfy the relation

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\Phi(x) - \Phi_0(x)|\, dx < \delta,$$

and

$$\Phi(0) < \Phi_0(0) + \delta$$

where $\delta > 0$. In case a $\Phi \in U$ satisfies the above two relations we shall write

$$||\Phi - \Phi_0||_m < \delta.$$

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[1]Absence of limits of integration will mean that the integral is taken over the interval $(-\infty, \infty)$.

C. The mean almost weak topology in $\mathcal{F}$. As a neighborhood basis of a $\Phi_0 \in \mathcal{F}$ we shall take those sets which for any δ satisfy simultaneously the relations in A and B. We shall designate such neighborhoods by

$$\mathfrak{M}_m[\{f_k\};\ \delta;\ \Phi_0].$$

D. The uniform topology in $\mathcal{F}$ and $\mathcal{M}$. Let us write

$$||\Phi - \Phi_0|| = \sup |\Phi(x) - \Phi_0(x)|,$$

where the sup is taken over all x on the real line. Then as a neighborhood basis of Φ_0 we shall take the sets which satisfy

$$||\Phi - \Phi_0|| < \delta.$$

The same type of topology on $\mathcal{M}$ will be called the uniform topology on $\mathcal{M}$.

E. The variational topology on $\mathcal{M}$. We shall write

$$||\phi - \phi_0||_v = \text{total variation } [\phi(t) - \phi_0(t)],$$

and as a neighborhood basis of ϕ_0 take the sets in $\mathcal{M}$ which satisfy

$$||\phi - \phi_0||_v < \delta.$$

Suppose now that $\phi \in \mathcal{M}$ and t a point where $\phi(t) - \phi(t-0) \geqslant \delta > 0$. Let $I(\phi; \delta; t)$ be a generic symbol for an open interval which contains the point t and let $\{I(\phi; \delta; t_k)\}$ represent a class of such intervals where the t_k run over all points for which the jump of $\phi(t)$ is greater than or equal to δ. Each such class of course contains only a finite number of members.

THEOREM 1. *Let $\Phi_0 \in \mathcal{F}$ and $\epsilon > 0$ be given. There exists a $\delta > 0$ such that if we exclude a small interval about each point of the real axis where the jump of $\phi_0(t)$ is greater than or equal to δ, we can find an almost weak neighborhood of Φ_0, $\mathfrak{M}[\{f_k\}; \delta; \Phi_0]$, so that outside the excluded intervals each element of $\mathcal{M}$ which corresponds to an element of $\mathfrak{M}[\{f_k\}; \delta; \Phi_0]$ is uniformly within ϵ of $\phi_0(t)$.*

In more technical language the above theorem can be stated as follows: *Given $\Phi_0 \in \mathcal{F}$ and $\epsilon > 0$. There exists a $\delta > 0$ such that for any $\{I(\phi_0; \delta; t_k)\}$ there exists an $\mathfrak{M}[\{f_k\}; \delta; \Phi_0]$ so that $\Phi \in \mathfrak{M}[\{f_k\}; \delta; \Phi_0]$ implies $|\phi(t) - \phi_0(t)| < \epsilon$ for all $t \notin \bigcup I(\phi_0; \delta; t_k)$.*

Proof. Let $\delta > 0$ be given and choose R sufficiently large so that

$$\int_{|t|>R} d\ \phi_0(t) < \delta$$

Further, choose $f_0^*(t)$ to be of class C^2 (continuous second derivatives) such that $0 \leqslant f_0^*(t) \leqslant 1$ and

$$f_0^*(t) = \begin{cases} 1, & |t| \leqslant R \\ 0, & |t| > R + 1. \end{cases}$$

Let

$$f_0(x) = \frac{1}{2\pi}\int e^{-ixt} f_0^*(t)\, dt.$$

Integrating by parts twice will immediately show that $f_0(x) \in L^1(-\infty, \infty)$. Further, since $f_0^*(t)$ itself belongs to $L^1(-\infty, \infty)$, is continuous and of bounded variation over the whole real axis, we have the inversion formula **(1, p. 42)**.

$$f_0^*(t) = \int f_0(x)\, e^{ixt} dx.$$

Therefore,

$$\int f_0^*(t)\, d\phi(t) = \int \left[\int f_0(x)\, e^{ixt}\, dx\right] d\phi(t).$$

Since $f_0(x) \in L^1(-\infty, \infty)$ and $\phi(t)$ is bounded we may apply Fubini's theorem **(4, p. 77)** and we get the Parseval relation

$$\int f_0^*(t)\, d\phi(t) = \int f_0(x)\, \Phi(x)\, dx,$$

Therefore, if we choose any Φ such that

$$\left| \int f_0(x)[\Phi(x) - \Phi_0(x)]\, dx \right| < \delta, \tag{1}$$

we have for the corresponding $\phi(t)$,

$$\left| \int f_0^*(t)\, d[\phi(t) - \phi_0(t)] \right| < \delta.$$

If Φ satisfies the further condition

$$\Phi(0) < \Phi_0(0) + \delta, \tag{2}$$

then we have

$$\Phi_0(0) + \delta > \Phi(0) \geqslant \int f_0^*(t)\, d\phi(t) > \int f_0^*(t)\, d\phi_0(t) - \delta > \Phi_0(0) - 2\delta. \tag{3}$$

Therefore,

$$0 \leqslant \int d\phi(t) - \int_{|t|<R+1} d\phi(t) \leqslant \int d\phi(t) - \int f_0^*(t)\, d\phi(t) < 3\delta,$$

from which we get

$$\phi(-R-1) < 3\delta. \tag{4}$$

Now, choose a set $\{I(\phi_0; \delta; t_k)\}$ and suppose there exists a t_0 in the complement of $\cup I(\phi_0; \delta; t_k)$ which lies to the right of $-R-1$. There exists an $h > 0$ such that

$$|\phi_0(t_0 \pm h) - \phi_0(t_0)| < \delta. \tag{5}$$

Choose $f_1^*(t)$ and $f_2^*(t)$ to be in C^2 with range in $[0, 1]$ and defined in the following way:

$$f_1^*(t) = \begin{cases} 1, & -R-1 < t < t_0 - h, \\ 0, & t < -R-2, \quad t > t_0, \end{cases}$$

$$f_2^*(t) = \begin{cases} 1, & -R-1 < t < t_0, \\ 0, & t < -R-2, \quad t > t_0 + h. \end{cases}$$

If $f_1(x)$ and $f_2(x)$ are the Fourier transforms respectively of $f_1^*(t)$ and $f_2^*(t)$, then f_1 and f_2 are in $L^1(-\infty, \infty)$.

Let $\Phi(x)$ be any element of $\mathfrak{F}$ which satisfies (1), (2) and the further conditions

$$\left| \int f_k(x)[\Phi(x) - \Phi_0(x)]\, dx \right| < \delta, \qquad k = 1, 2.$$

By the Parseval relation we have for $k = 1, 2$,

$$\left| \int f_k^*(t)\, d[\phi(t) - \phi_0(t)] \right| < \delta \tag{6}$$

Consequently, by (4), (5) and (6) we get

$$\phi_0(t_0) - 3\delta < \int f_1^*(t)\, d\phi(t) < \phi(t_0),$$

$$\phi(t_0) - 3\delta < \int f_2^*(t)\, d\phi(t) < \phi_0(t_0) + 2\delta.$$

From this it follows that

$$-5\delta < \phi_0(t_0) - \phi(t_0) < 3\delta.$$

The complement of $\cup I(\phi_0; \delta; t_k)$ (which we may as well suppose is not the null set) which lies in the interval $(-R-1, \infty)$ consists of a finite number of mutually disjoint intervals. In each such interval it is possible to find a finite set of numbers $\tau_1 < \tau_2 < \ldots < \tau_n$ such that τ_1 and τ_n are the endpoints of the interval and

$$\phi_0(\tau_{k+1}) - \phi_0(\tau_k) < \delta.$$

Therefore, there exist functions $\{f_k(x)\}$ each of which belongs to $L^1(-\infty, \infty)$ such that if $\Phi(x) \in \mathfrak{M}[\{f_k\}; \delta; \Phi_0]$ we have

$$|\phi(\tau_k) - \phi_0(\tau_k)| < 5\delta.$$

((2) and (3) also give us this relation for $\tau_k = \infty$.)

Suppose $\tau_k \leqslant t \leqslant \tau_{k+1}$. Then

$$\phi_0(\tau_k) \leqslant \phi_0(t) \leqslant \phi_0(\tau_{k+1}), \quad \phi(\tau_k) \leqslant \phi(t) \leqslant \phi(\tau_{k+1}).$$

Therefore

$$-6\delta < \phi(\tau_k) - \phi_0(\tau_{k+1}) \leqslant \phi(t) - \phi_0(t) \leqslant \phi(\tau_{k+1}) - \phi_0(\tau_k) < 6\delta. \tag{7}$$

Since we are dealing with only a finite number of intervals in the complement of $\cup I(\phi_0; \delta; t_k)$ which lies in $(-R-1, \infty)$ we can find an almost weak

neighborhood of Φ_0 such that if Φ belongs to this neighborhood, then the corresponding functions satisfy (7). If we now choose $\delta = \frac{1}{2}\epsilon$ we have our theorem.

COROLLARY. *If $\phi_0(t)$ is continuous then the mapping from $\mathcal{F}$, with the almost weak topology, to $\mathcal{M}$, with the uniform topology, is continuous at Φ_0.*

THEOREM 2. *Let $\phi_0(t) \in \mathcal{M}$ be a step function. Then given $\epsilon > 0$, there exists a $\delta > 0$ such that*

$$||\Phi - \Phi_0||_m < \delta$$

implies

$$||\phi - \phi_0||_s < \epsilon.$$

Proof. Given $\phi(t)$ and $\phi_0(t)$, let t_n be the set of points where either $\phi(t)$ or $\phi_0(t)$ has a jump. Let a_n and b_n be respectively the jump of $\phi_0(t)$ and $\phi(t)$ at t_n. Let us write

$$\phi(t) = S(t) + D(t),$$

where $S(t)$ is a step function and $D(t)$ is a continuous function. We then have

$$\phi(t) - \phi_0(t) = \{S(t) - \phi_0(t)\} + D(t).$$

Since $\phi_0(t)$ is a step function, $S(t) - \phi_0(t)$ is either a step function or identically zero since $S(-\infty) = \phi_0(-\infty) = 0$. This gives us the decomposition of $\phi(t) - \phi_0(t)$ into a step function and a continuous function. Therefore **(2, pp. 189–190)**

$$||\phi - \phi_0||_s = ||S - \phi_0||_s + ||D||_s$$

Now, let $\psi(t) = S(t) - \phi_0(t)$. Then **(2, pp. 188–190)**,

$$||S - \phi_0||_s = ||\psi||_s = \sum_{n=1}^{\infty} \{|\psi(t_n + 0) - \psi(t_n)| + |\psi(t_n) - \psi(t_n - 0)|\}$$

By normalization of the functions in $\mathcal{M}$ we have

$$|\psi(t_n + 0) - \psi(t_n)| = 0$$

Therefore

$$||S - \phi_0||_s = \sum_{n=1}^{\infty} |b_n - a_n|.$$

Consequently

$$\begin{aligned} ||\phi - \phi_0||_s &= \sum_{n=1}^{\infty} |b_n - a_n| + ||D||_s \\ &< \sum_{n=1}^{N} |b_n - a_n| + ||D||_s + \sum_{n=N+1}^{\infty} b_n + \sum_{n=N+1}^{\infty} a_n. \end{aligned}$$

Since

$$\Phi(0) = ||\phi||_s = \sum_{n=1}^{\infty} b_n + ||D||_s,$$

we have

$$||\phi - \phi_0||_* = \sum_{n=1}^{N} |b_n - a_n| + \Phi(0) - \sum_{n=1}^{N} b_n + \sum_{n=N+1}^{\infty} a_n.$$

Let us here make the parenthetical remark that if either $\phi(t)$ or $\phi_0(t)$ has a finite number of jumps, then b_n or a_n from some point on will be zero.

Now,

$$\Phi(0) - \sum_{n=1}^{N} b_n = \Phi(0) - \Phi_0(0) + \sum_{n=N+1}^{\infty} a_n - \sum_{n=1}^{N} (b_n - a_n).$$

Therefore

$$||\phi - \phi_0||_* \leqslant 2\sum_{n=1}^{N} |b_n - a_n| + 2\sum_{n=N+1}^{\infty} a_n + \Phi(0) - \Phi_0(0).$$

Choose N so that

$$\sum_{n=N+1}^{\infty} a_n < \epsilon/5$$

and then choose $\delta \leqslant \epsilon/5N$. It is well known (**1**, p. 79, Satz 24) that

$$a_n = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-it_n x} \phi_0(x)\, dx,$$

$$b_n = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} e^{-it_n x} \phi(x)\, dx.$$

Therefore

$$|b_n - a_n| \leqslant \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\phi(x) - \phi_0(x)|\, dx.$$

From this inequality we get the desired result.

From the two preceding results we might expect that if $\mathscr{F}$ is given the mean almost weak topology and $\mathscr{M}$ the uniform topology, then the mapping from $\mathscr{F}$ to $\mathscr{M}$ is continuous. This is shown by the next theorem.

THEOREM 3. *Given* $\Phi_0 \in \mathscr{F}$ *and* $\epsilon > 0$, *there exists a neighborhood* $\mathfrak{M}_m(\{f_k\}; \delta; \Phi_0)$ *such that* $\Phi \in \mathfrak{M}_m$ implies

$$||\phi - \phi_0|| < \epsilon.$$

Proof. As in the proof of Theorem 1, let $\delta > 0$ be given and choose R sufficiently large so that

$$\int_{|t|>R} d\phi_0(t) < \delta.$$

Also, choose $f_0^*(t)$ as in Theorem 1 and let $f_0(x)$ be its Fourier transform. Then if $\Phi \in \mathscr{F}$ is such that

$$\Phi(0) \leqslant \Phi_0(0) + \delta \tag{2}$$

and

$$\left| \int f_0(x)[\Phi(x) - \Phi_0(x)]\, dx \right| < \delta;$$

as in Theorem 1 we get, for $t < -R - 1$,

$$0 \leqslant \phi(t) \leqslant \phi(-R - 1) < 3\delta,$$

and

$$\Phi(0) > \Phi_0(0) - 2\delta.$$

Suppose now that $\{\tau_k\}$ is the finite set of points to the right of $-R - 1$ for which $\phi_0(\tau_k) - \phi_0(\tau_k - 0) \geqslant \delta$. The interval $[\tau_k, \tau_{k+1}]$ may be subdivided by a finite number of points

$$\tau_k = \tau_{0,k} < \tau_{1,k} < \ldots < \tau_{m,k} = \tau_{k+1}$$

such that

$$\phi_0(\tau_{j+1,k}) - \phi_0(\tau_{j,k}) < \delta, \qquad j = 0, 1, \ldots, m - 1,$$

and

$$\phi_0(\tau_{k+1} - 0) - \phi_0(\tau_{m-1,k}) < \delta.$$

Therefore, there exists a finite set of points, $-R - 1 = t_0 < t_1 < \ldots < t_n = \infty$, which includes the set $\{\tau_k\}$ and such that

$$\phi_0(t_{k+1}) - \phi_0(t_k) < \delta, \qquad t_{k+1} \notin \{\tau_k\},$$

and

$$\phi_0(t_{k+1} - 0) - \phi_0(t_k) < \delta, \qquad t_{k+1} \in \{\tau_k\}.$$

For $k = 1, \ldots, n - 2$, choose, as in Theorem 1, $f_k^*(t) \in C^2$ and with range in $[0, 1]$ in the following manner:

$$f_k^*(t) = \begin{cases} 1, & t_0 \leqslant t \leqslant t_k \\ 0, & t > t_{k+1}, \quad t < t_0 - 1. \end{cases}$$

Further, choose $f_{n-1}^*(t) \in C^2$ such that $0 \leqslant f_{n-1}^*(t) \leqslant 1$ and

$$f_{n-1}^*(t) = \begin{cases} 1, & t_0 \leqslant t \leqslant t_{n-1} \\ 0, & t > t_{n-1} + 1, \quad t < t_0 - 1. \end{cases}$$

Let $f_k(x)$ be the Fourier transform of $f_k^*(t)$. Then if we choose Φ to satisfy (2) and

$$\left| \int f_k(x)[\Phi(x) - \Phi_0(x)]\, dx \right| < \delta, \qquad k = 0, 1, \ldots, n - 1, \tag{8}$$

then for $t_k \notin \{\tau_k\}$, by the same method of proof as in Theorem 1 we have

$$|\phi(t_k) - \phi_0(t_k)| < 5\delta.$$

If $t_k \in \{\tau_k\}$ then we have

$$\phi_0(t_k - 0) - 3\delta < \int f_{k-1}^*(t)\, d\phi(t) \leqslant \phi(t_k - 0),$$

from which

$$\phi_0(t_k - 0) - \phi(t_k - 0) < 3\delta.$$

Further, for the same t_k

$$\phi(t_k) - \phi(-R-1) \leqslant \int f_k^*(t)\, d\phi(t) < \phi_0(t_k) + 2\delta,$$

from which

$$\phi_0(t_k) - \phi(t_k) > -5\delta.$$

In addition to (2) and (8) let us now pick $\Phi \in U$ to also satisfy

$$\lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\Phi(x) - \Phi_0(x)|\, dx < \delta. \tag{9}$$

Suppose

$$\phi_0(t_k - 0) - \phi(t_k - 0) \leqslant -6\delta \text{ or } \phi_0(t_k) - \phi(t_k) \geqslant 5\delta.$$

Then, if a_k and b_k are respectively the jump of $\phi_0(t)$ and $\phi(t)$ at t_k we have

$$a_k - b_k \geqslant \delta.$$

But since

$$|a_k - b_k| \leqslant \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} |\Phi(x) - \Phi_0(x)|\, dx < \delta,$$

we get a contradiction. Therefore,

$$|\phi_0(t_k - 0) - \phi(t_k - 0)| < 6\delta,$$

and

$$|\phi_0(t_k) - \phi(t_k)| < 5\delta.$$

If we now proceed as in Theorem 1, the proof of our theorem is complete.

From this theorem we get the following corollary, which was originally proved by Dyson **(3)**.

COROLLARY. *Given $\Phi_0 \in \mathscr{F}$ and $\epsilon > 0$, there exists a $\delta_1 > 0$ such that*

$$||\Phi - \Phi_0|| < \delta_1 \textit{ implies } ||\phi - \phi_0|| < \epsilon.$$

Proof. Let

$$M = \max_k \int |f_k(x)|\, dx,$$

where $\{f_k\}$ is the set in Theorem 3. Then choose $\delta_1 = \delta/M$, where δ is that of Theorem 3.

In closing this paper we wish to remark that if we replace the space $\mathscr{M}$ by the space $\mathscr{B}$ of all functions of total bounded variation defined on the line and normalized in the same way as in $\mathscr{M}$, then our previous theorems can be given a meaning. We shall write down these corresponding theorems without proof and only remark that the proofs follow the pattern we have established before with only some slight modification.

THEOREM 1′. *Let a continuous* $\phi_0 \in \mathcal{B}$ *and* $\epsilon > 0$ *be given. Then there exists a* $\delta > 0$ *and functions* $\{f_k\}_1^n \subset C^2$ *such that*

$$\left| \int f_k \, d[\phi - \phi_0] \right| < \delta,$$

and

$$||\phi||_v \leqslant ||\phi_0||_v + \delta$$

implies

$$||\phi - \phi_0|| < \epsilon.$$

THEOREM 2′. *Let* $\phi_0 \in \mathcal{B}$ *be a step function. Then given* $\epsilon > 0$, *there exists a* $\delta > 0$ *such that*

$$\max |\text{saltus } [\phi(t) - \phi_0(t)]| < \delta$$

and

$$||\phi||_v \leqslant ||\phi_0||_v + \delta$$

implies

$$||\phi - \phi_0||_v < \epsilon.$$

THEOREM 3′. *Let* $\phi_0 \in \mathcal{B}$ *and* $\epsilon > 0$ *be given. Then there exist a* $\delta > 0$ *and* $\{f_k\}_1^n \subset C^2$ *such that*

$$\left| \int f_k \, d[\phi - \phi_0] \right| < \delta, \quad \max |\text{saltus}[\phi(t) - \phi_0(t)]| < \delta,$$

and

$$||\phi||_v \leqslant ||\phi_0||_v + \delta$$

implies

$$||\phi - \phi_0|| < \epsilon.$$

In the above theorems it is of course understood that ϕ belongs to $\mathcal{B}$.

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WEIGHTED QUADRATIC NORMS AND LEGENDRE POLYNOMIALS

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1. Introduction. Let $\omega_n(x) = (n + \frac{1}{2})^{\frac{1}{2}} P_n(x)$, $n = 0, 1, \ldots$, be the normalized Legendre polynomials. If $f(x) \in L^1(-1, 1)$ and if

$$a_n = \int_{-1}^{1} \omega_n(x) f(x)\, dx$$

then we write

$$f(x) \sim \sum_0^\infty a_n \omega_n(x).$$

Let $T = \{t_n\}$ $(0 \leqslant n < \infty)$ be a sequence of real constants. T determines a linear transformation on setting $Tf(x) = g(x)$ if $g(x) \in L^1(-1, 1)$ and if

$$g(x) \sim \sum_0^\infty t_n a_n \omega_n(x).$$

(In general T will not be defined for every $f \in L^1(-1, 1)$). Let

$$\mathfrak{N}_{\alpha,\beta}[f] = \left[\int_{-1}^{1} (1 + x)^\alpha (1 - x)^\beta [f(x)]^2 dx \right]^{\frac{1}{2}}$$

where $-1 < \alpha, \beta < 1$. $\mathfrak{N}_{\alpha,\beta}$ will also be used to denote the space of functions $f(x)$ for which $\mathfrak{N}_{\alpha,\beta}[f]$ is finite. Because of the special role played by $x = \pm 1$ in Legendre series such norms are quite natural. Our objective in the present paper is to give a rather general sufficient condition for T to be a bounded linear transformation of $\mathfrak{N}_{\alpha,\beta}$ into itself.

The origin of this problem lies in the fact that T is a multiplier transformation and multiplier transformations for Fourier series have been extensively studied. Let, just as above, $T = \{t_n\}$ $(-\infty < n < \infty)$ be formally defined as that linear transformation which carries

$$f(\theta) \sim \sum_{-\infty}^{\infty} a_n e^{in\theta}$$

into

$$Tf(\theta) \sim \sum_{-\infty}^{\infty} t_n a_n e^{in\theta}.$$

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Let

$$||f(\theta)||_p = \left[\int_{-\pi}^{\pi} |f(\theta)|^p d\theta\right]^{1/p}.$$

T is said to be of type $[p, q]$ if there exists a constant A such that $||Tf||_q \leqslant A||f||_p$ for every f for which $||f||_p$ is finite. Many classical investigations in the theory of Fourier series have been concerned with the problem of showing that a particular multiplier transformation is of some prescribed type. Thus the conjugate function theorem of M. Riesz asserts that $\{i \operatorname{sgn} n\}$ is of type $[p, p]$ for $1 < p < \infty$, and the fractional integration theorem of Hardy and Littlewood is equivalent to the assertion that $\{(in)^{-\sigma}\}'$ is of type $[p, p(1 - p\sigma)^{-1}]$ for $1 < p < \sigma^{-1} < \infty$. (Here the prime indicates that the term corresponding to $n = 0$ is omitted.) An investigation of particular relevance to the present paper is that of Marcinkiewicz **(4)** who, by making use of some very difficult researches of Paley, Littlewood, and later Zygmund, was able to prove that T is of type $[p, p]$, $1 < p < \infty$, if

1.1
$$|t_n| \leqslant A \qquad (n = 0, \pm 1, \pm 2, \ldots),$$

$$\sum_{\pm 2^n}^{\pm 2^{n+1}} |t_k - t_{k-1}| \leqslant A \qquad (n = 0, 1, 2, \ldots).$$

Marcinkiewicz's result plays a central role in the theory of multiplier transformations in that many of the other results can be deduced from it. Let

$$||f||_{\alpha,p} = \left[\int_{-\pi}^{\pi} |f(\theta)|^p |\theta|^{\alpha p} d\theta\right]^{1/p}.$$

T will be said to be of type $[(\alpha, p), (\beta, q)]$ if there exists a constant A such that $||Tf||_{\beta,q} \leqslant A||f||_{\alpha,p}$ for all $f(\theta)$ for which $||f||_{\alpha,p}$ is finite. The author has recently proved that if the conditions (1) hold then T is of type $[(\alpha, p), (\alpha, p)]$ for $1 < p < \infty$, $-p^{-1} < \alpha < 1 - p^{-1}$. See **(3)**. The arguments used depend heavily upon the connection between Fourier series and power series, and thus cannot be extended to other orthogonal expansions. However it can be shown that the conditions 1.1 imply that T is of type $[(\alpha, 2), (\alpha, 2)]$ by quite different arguments. This is, of course, the case of weighted quadratic norms. In the present paper we shall apply these arguments to Legendre series. It will be convenient to state our principal result.

DEFINITION. $T = \{t_n\}$ $(0 \leqslant n < \infty)$ *is said to belong to class* $\mathbf{M}(C)$ *if*:

(a)
$$|t_n| \leqslant C \qquad (n = 0, 1, 2, \ldots);$$

(b)
$$\sum_{2^n}^{2^{n+1}} |t_k - t_{k-1}| \leqslant C \qquad (n = 0, 1, 2, \ldots).$$

We shall show that if T belongs to class $\mathbf{M}(C)$ then

$$\mathfrak{N}_{\alpha,\beta}[Tf] \leqslant D(\alpha, \beta) C \mathfrak{N}_{\alpha,\beta}[f] \qquad (-\tfrac{1}{2} < \alpha, \beta < \tfrac{1}{2})$$

where $D(\alpha, \beta)$ depends only upon α and β. The restriction $-\frac{1}{2} < \alpha, \beta < \frac{1}{2}$

here is essential and the result is not otherwise true. As an example let $S_N = \{s_{N,n}\}$ where $s_{N,n}$ is 1 for $0 \leqslant n \leqslant N$ and is 0 for $N < n < \infty$. S_N belongs to $\mathbf{M}(1)$. Thus

$$\mathfrak{N}_{\alpha,\beta}[S_N f] \leqslant D(\alpha,\beta)\mathfrak{N}_{\alpha,\beta}[f] \qquad (-\tfrac{1}{2} < \alpha, \beta < \tfrac{1}{2}). \tag{1.2}$$

For the sake of comparison we recall that Pollard has shown in **(6)** that

$$||S_N f||_p \leqslant D(p)||f||_p \qquad (\tfrac{4}{3} < p < 4),$$

the result being false for other values of p. See also in this connection the paper of Newman and Rudin **(5)**.

Our method of demonstration depends upon the following inequalities, valid for $0 < \alpha < \frac{1}{2}$,

$$E'(\alpha) \leqslant \mathfrak{N}_{0,\alpha}[f]^{-2} \sum_{k>j} [a_j(j+\tfrac{1}{2})^{-\frac{1}{2}} - a_k(k+\tfrac{1}{2})^{-\frac{1}{2}}]^2 Q(k,j) \leqslant E''(\alpha),$$

$$E'(\alpha) \leqslant \mathfrak{N}_{\alpha,0}[f]^{-2} \sum_{k>j} [(-1)^j a_j(j+\tfrac{1}{2})^{-\frac{1}{2}} - (-1)^k a_k(k+\tfrac{1}{2})^{-\frac{1}{2}}]^2 Q(k,j) \leqslant E''(\alpha),$$

where

$$Q(k,j) = (k+\tfrac{1}{2})(j+\tfrac{1}{2})|k-j|^{-1-2\alpha}(k+j)^{-1}.$$

Here $E'(\alpha)$ and $E''(\alpha)$ are positive constants depending only upon α.

2. Preliminary estimates. The present section is devoted to establishing results which are needed to prove the identities announced at the end of §1. Let us write $\phi(x) \simeq \psi(x)$ $(x \in X)$ if there exist finite positive constants c_1 and c_2 such that $c_1 \leqslant \phi(x)/\psi(x) \leqslant c_2$ $(x \in X)$. Similarly $\phi(x) \leqslant \simeq \psi(x)$ $(x \in X)$ if there exists a positive constant c such that $\phi(x) \leqslant c\psi(x)$ for $(x \in X)$. We define

$$\Omega_\alpha(x) = \sum_{n=1}^{\infty} [1 - P_n(x)]n^{-1-2\alpha}.$$

LEMMA 2a. *For α fixed $0 < \alpha < \frac{1}{2}$ we have*

$$\Omega_\alpha(x) \simeq (1-x)^\alpha \qquad (-1 \leqslant x \leqslant 1).$$

Proof. Let $g_0 = 1$,

$$g_m = \frac{1.3 \ldots (2m-1)}{2.4 \ldots 2m};$$

then **(8**, p. 92**)**

$$P_n(\cos\theta) = \sum_{m=0}^{n} g_m g_{n-m} \cos(n-2m)\theta.$$

Setting $\theta = 0$ we see that

$$1 = \sum_{m=0}^{n} g_m g_{n-m},$$

and thus

$$1 - P_n(\cos\theta) = \sum_{m=0}^{n} g_m g_{n-m}[1 - \cos(n-2m)\theta].$$

Distinguishing between even and odd, we find

$$\sum_{n=1}^{\infty}[1-P_{2n}(\cos\theta)](2n)^{-1-2\alpha}=2\sum_{j=1}^{\infty}[1-\cos(2j)\theta]\sum_{n>j}(2n)^{-1-2\alpha}g_{n-j}g_{n+j},$$

$$\sum_{n=0}^{\infty}[1-P_{2n+1}(\cos\theta)](2n+1)^{-1-2\alpha}$$

$$=2\sum_{j=0}^{\infty}[1-\cos(2j+1)\theta]\sum_{n>j}(2n+1)^{-1-2\alpha}g_{n-j}g_{n+1+j}.$$

Since

$$g_m(\pi m)^{\frac{1}{2}}\to 1 \qquad (m\to\infty)$$

it is easy to establish that

$$\sum_{n>j}(2n)^{-1-2\alpha}g_{n-j}g_{n+j}\simeq(2j)^{-1-2\alpha},$$

$$\sum_{n>j}(2n+1)^{-1-2\alpha}g_{n-j}g_{n+j+1}\simeq(2j+1)^{-1-2\alpha},$$

and thus that

$$\sum_{n=1}^{\infty}[1-P_n(\cos\theta)]n^{-1-2\alpha}\simeq\sum_{j=1}^{\infty}[1-\cos(j\theta)]\,j^{-1-2\alpha}.$$

Let

$$p(\theta)=\sum_{j=1}^{\infty}[1-\cos(j\theta)]\,j^{-1-2\alpha};$$

then

$$p'(\theta)=\sum_{j=1}^{\infty}[\sin(j\theta)]\,j^{-2\alpha}.$$

It follows (**10**, pp. 112–116) that $p'(\theta)\simeq\theta^{2\alpha-1}$ $(\theta\to 0+)$ and thus that $p(\theta)\simeq\theta^{2\alpha}(0<\theta<\pi)$. Combining these results we have the conclusion of our lemma.

LEMMA 2b. *If α is fixed, $(0<\alpha<\frac{1}{2})$, then*

$$\int_{-1}^{1}\Omega_\alpha(x)\,P_k(x)\,P_j(x)\,dx\simeq-(k-j)^{-1-2\alpha}(k+j)^{-1} \qquad (k>j\geqslant 0).$$

Proof. The integral

$$\int_{-1}^{1}P_k(x)\,P_j(x)\,P_r(x)\,dx \qquad (k>j)$$

is zero except for those values of r such that $k-j<r<k+j$ and such that the parity of r equals the parity of $j+k$. In these cases the value of the integral is given by the formula (**1**, p. 311)

$$\int_{-1}^{1}P_k(x)\,P_j(x)\,P_{k+j-2s}(x)\,dx=\frac{2g_{k-s}g_{j-s}g_s}{(2k+2j-s+1)g_{k+j-s}}.$$

Thus

$$\int_{-1}^{1} \Omega_\alpha(x)\, P_k(x)\, P_j(x)\, dx = -\, S(k,j),$$

where

$$S(k,j) = \sum_{s=0}^{j} \frac{2g_{k-s}g_{j-s}g_s}{(2k+2j-s+1)g_{k+j-s}} (k+j-2s)^{-1-2\alpha}.$$

We have

$$S(k,j) \simeq k^{-1} \sum_{s=0}^{j} (k+1-s)^{-\frac{1}{2}}(j+1-s)^{-\frac{1}{2}}(s+1)^{-\frac{1}{2}}(k+j-2s)^{-1-2\alpha}.$$

If $k > 3j/2$, $0 \leqslant s \leqslant j$, then $(k+1-s) \simeq k$, $(k+j-2s) \simeq k$, so that

$$S(k,j) \simeq k^{-2-2\alpha} \sum_{s=0}^{j} (j+1-s)^{-\frac{1}{2}}(s+1)^{-\frac{1}{2}} \simeq k^{-2-2\alpha},$$

which can be rewritten as

$$S(k,j) \simeq (k-j)^{-1-2\alpha}(k+j)^{-1} \qquad (k > 3j/2).$$

On the other hand if $j < k < 3j/2$ then

$$S(k,j) = S_1(k,j) + S_2(k,j) + S_3(k,j) = \sum_{2j-k \leqslant s \leqslant j} + \sum_{\frac{1}{2}j < s < 2j-k} + \sum_{0 \leqslant s < \frac{1}{2}j}.$$

If $j < k < 3j/2$ and if $2j-k \leqslant s \leqslant j$ then $(k+1-s) \simeq (k-j)$, $(k+j-2s) \simeq (k-j)$, and $(s+1) \simeq k$, so that

$$S_1(k,j) \simeq k^{-1}(k-j)^{-\frac{1}{2}-1-2\alpha} \sum_{2j-k \leqslant s \leqslant j} (j+1-s)^{-\frac{1}{2}}$$
$$\simeq (k-j)^{-1-2\alpha}(k+j)^{-1}.$$

If $j < k < 3j/2$ and $j/2 \leqslant s < 2j-k$ then $(k+1-s) \simeq (k-s)$, $(k+j-2s) \simeq (k-s)$, $(s+1) \simeq k$, and $(j+1-s) \simeq (k-s)$, so that

$$S_2(k,j) \simeq k^{-1} \sum_{\frac{1}{2}j < s < 2j-k} (k-s)^{-2-2\alpha} \simeq (k-j)^{-1-2\alpha}(k+j).$$

Finally if $j < k < 3j/2$, $0 \leqslant s \leqslant \frac{1}{2}j$, then $(k+1-s) \simeq k$, $(j+1-s) \simeq k$, and $(k+j-2s) \simeq k$, so that

$$S_3(k,j) \simeq k^{-5/2-2\alpha} \sum_{0 \leqslant s < \frac{1}{2}j} (s+1)^{-\frac{1}{2}} < \simeq (k-j)^{-1-2\alpha}(k+j)^{-1}.$$

It follows that

$$S(k,j) \simeq (k-j)^{-1-2\alpha}(k+j)^{-1} \qquad (j < k < 3j/2),$$

and our demonstration is complete.

Let us set

$$\int_{-1}^{1} \Omega_\alpha(x)(k+\tfrac{1}{2})\, P_k(x)(j+\tfrac{1}{2})\, P_j(x)\, dx = \begin{cases} d_k, & k = j, \\ -c_{kj}, & k \neq j. \end{cases}$$

Since $\Omega_\alpha > 0$ it is clear that $d_k > 0$, while $c_{kj} > 0$ by Lemma 2b.

LEMMA 2c.

$$\sum_{\substack{k=0 \\ k\neq j}}^{\infty} c_{kj} = d_j.$$

Proof. We have (9, p. 332).

$$p(h,x) = \sum_{k=0}^{\infty} h^k(k+\tfrac{1}{2})\,P_k(x) = \frac{\frac{1}{2}(1-h^2)}{(1-2hx+h^2)^{3/2}}.$$

If $-1 < h < 1$ then

$$\int_{-1}^{1} \Omega_n(x)(j+\tfrac{1}{2})\,P_j(x)\,p(h,x)\,dx = d_j h^j - \sum_{\substack{k=0 \\ k\neq j}}^{\infty} c_{kj}h^k.$$

It is easily verified, by integrating term by term, that

$$\int_{-1}^{1} p(h,x)\,dx = 1, \qquad -1 < h < 1,$$

and it is easy to see that

$$\lim_{h\to 1-} p(h,x) = 0$$

uniformly for $-1 \leqslant x \leqslant 1-\epsilon$ for any fixed $\epsilon > 0$. From these relations we have

$$\lim_{h\to 1-} \int_{-1}^{1} \Omega_n(x)(j+\tfrac{1}{2})\,P_j(x)\,p(h,x)\,dx = \Omega_n(1)(j+\tfrac{1}{2})\,P_j(1) = 0;$$

that is,

$$\lim_{h\to 1-} \sum_{\substack{k=0 \\ k\neq j}}^{\infty} c_{kj}h^k = d_j,$$

and since the c_{kj} are positive the series on the left is not only Abel summable but actually convergent.

3. Some inequalities. The following result is demonstrated in (2):

THEOREM 3a. *If $\phi_n(\theta)$ $n = 0, 1, \ldots$ is a uniformly bounded orthonormal set on $0 \leqslant \theta \leqslant 1$ and if for $F(\theta) \in L^1(0,1)$*

$$a_n = \int_0^1 F(\theta)\,\phi_n(\theta)\,d\theta,$$

then

$$\sum_0^{\infty} a_k^2(n_k+1)^{-2\alpha} \leqslant A(\alpha)\int_0^1 F(\theta)^2\theta^{2\alpha}d\theta \qquad (0 \leqslant \alpha < \tfrac{1}{2}),$$

where $n_0, n_1, n_2, \ldots$ is any rearrangement of $0, 1, 2, \ldots$.

In order to apply this theorem to Legendre polynomials let us define

$$\phi_n(\theta) = \omega_n(\cos \pi\theta)[\pi \sin \pi\theta]^{\frac{1}{2}}$$
$$F(\theta) = f(\cos \pi\theta)[\pi \sin \pi\theta]^{\frac{1}{2}}$$

$(0 \leqslant \theta \leqslant 1)$. Setting $x = \cos \pi\theta$ we see that

$$a_n = \int_{-1}^{1} f(x)\, \omega_n(x)\, dx = \int_0^1 F(\theta)\, \phi_n(\theta)\, d\theta.$$

The functions $\phi_n(\theta)$ are evidently orthonormal, that they are uniformly bounded follows from (8; p. 161). Thus

$$\sum_0^\infty a_k^2 (n_k + 1)^{-2\alpha} \leqslant A(\alpha) \int_0^1 F(\theta)^2 \theta^{2\alpha} d\theta.$$

We have

$$\int_0^1 F(\theta)^2 \theta^{2\alpha} d\theta = \int_{-1}^{1} f(x)^2 \left[\frac{1}{\pi} \arccos x\right]^{2\alpha} dx \simeq \mathfrak{N}_{0,\alpha}[f]^2.$$

Combining these results we have proved (7)

THEOREM 3b. *If* $0 \leqslant \alpha < \frac{1}{2}$ *then*

$$\sum_0^\infty a_k^2 (n_k + 1)^{-2\alpha} \leqslant B(\alpha)\, \mathfrak{N}_{0,\alpha}[f]^2,$$

where $n_0, n_1, n_2, \ldots$ *is any rearrangement of* $0, 1, 2, \ldots$.

We proceed to prove

LEMMA 3c. *If* $0 \leqslant \alpha < \frac{1}{2}$ *and if*

(i) $$f(x) \in \mathfrak{N}_{0,\alpha},$$

(ii) $$f(x) \sim \sum_0^\infty a_n \omega_n(x),$$

then

$$\lim_{N \to \infty} \sum_0^N a_n^2 (N - n + 1)^{-2\alpha} = 0.$$

Proof. Given $\epsilon > 0$, let us choose a finite sum

$$g(x) = \sum_0^M b_n \omega_n(x)$$

such that $\mathfrak{N}_{0,\alpha}[f - g]^2 \leqslant \epsilon$. By Theorem 3b,

$$\sum_0^N (a_n - b_n)^2 (N - n + 1)^{-2\alpha} \leqslant A\, \mathfrak{N}_{0,\alpha}[f - g]^2 \leqslant A\epsilon$$

where b_n is defined as 0 if $n > M$. Now

$$\sum_0^N a_n^2 (N - n + 1)^{-2\alpha} = \sum_{M+1}^N (a_n - b_n)^2 (N - n + 1)^{-2\alpha} + \sum_{n=1}^M a_n^2 (N - n + 1)^{-2\alpha}.$$

Thus

$$\limsup_{N\to\infty} \sum_{0}^{N} a_n^2 (N-n+1)^{-2\alpha} \leqslant A\epsilon.$$

Since ϵ is arbitrary our desired conclusion follows.

4. The basic norm relations. We need the following version of the Riesz Fischer theorem, the proof of which is omitted:

LEMMA 4a. *Let $\alpha(-1 < \alpha < 1)$ be fixed. If constants $\{a_n\}_0^\infty$ are given and if*

$$\liminf_{r\to\infty} \mathfrak{N}_{0,\alpha}\Big[\sum_{0}^{r} a_n\omega_n(x)\Big] \leqslant A,$$

then there exists a (unique) function $f(x)$ such that

$$\mathfrak{N}_{0,\alpha}[f] \leqslant A, \quad f(x) \sim \sum_{0}^{\infty} a_n\omega_n(x).$$

THEOREM 4b. *If, for $0 < \alpha < \frac{1}{2}$, α fixed*

(i) $$f(x) \in \mathfrak{N}_{0,\alpha},$$

(ii) $$f(x) \sim \sum_{0}^{\infty} a_n\omega_n(x),$$

then

$$\mathfrak{N}_{0,\alpha}[f]^2 \simeq \sum_{k>j} [a_k(k+\tfrac{1}{2})^{-\frac{1}{2}} - a_j(j+\tfrac{1}{2})^{-\frac{1}{2}}]^2 Q(k,j).$$

Proof. Let

$$g(x) = \sum_{0}^{n} b_k\omega_k(x);$$

then by Lemma 2a,

$$\mathfrak{N}_{0,\alpha}[g]^2 \simeq \int_{-1}^{1} \Omega_\alpha(x)[g(x)]^2 dx.$$

Using Lemma 2c we see that

$$\begin{aligned}\int_{-1}^{1} \Omega_\alpha(x)[g(x)]^2 dx &= \sum_{k=0}^{n} d_k b_k^2 (k+\tfrac{1}{2})^{-1} - \sum_{j,k=0,\ j\neq k}^{n} c_{kj} b_k (k+\tfrac{1}{2})^{-\frac{1}{2}} b_j (j+\tfrac{1}{2})^{-\frac{1}{2}}, \\ &= \sum_{k>j} [b_k(k+\tfrac{1}{2})^{-\frac{1}{2}} - b_j(j+\tfrac{1}{2})^{-\frac{1}{2}}]^2 c_{kj},\end{aligned}$$

where in this sum k and j vary from 0 to ∞, b_k being equal to zero, for $k > n$. Applying Lemma 2b we have

$$\mathfrak{N}_{0,\alpha}[g]^2 \simeq \sum_{k>j} [b_k(k+\tfrac{1}{2})^{-\frac{1}{2}} - b_j(j+\tfrac{1}{2})^{-\frac{1}{2}}]^2 Q(k,j).$$

Thus our theorem is true for finite sums of Legendre polynomials.

Let $f(x) \in \mathfrak{N}_{0,\alpha}$ and let

$$h_r(x) = \sum_{k=0}^{r} a_{r,k}\omega_k(x)$$

be a series of polynomials such that

$$\lim_{r\to\infty} \mathfrak{N}_{0,\alpha}[f - h_r] = 0.$$

Note that this implies that

$$\lim_{r\to\infty} a_{r,k} = a_k \qquad (k = 0, 1, \ldots).$$

We have, if N is fixed,

$$\sum_{0<j<k<N} [a_{r,k}(k + \tfrac{1}{2})^{-\frac{1}{4}} - a_{r,j}(j + \tfrac{1}{2})^{-\frac{1}{4}}]^2 Q(k,j) < A\, \mathfrak{N}_{0,\alpha}[h_r]^2.$$

Letting r increase without limit, we obtain

$$\sum_{0<j<k<N} [a_k(k + \tfrac{1}{2})^{-\frac{1}{4}} - a_j(j + \tfrac{1}{2})^{-\frac{1}{4}}]^2 Q(k,j) < A\, \mathfrak{N}_{0,\alpha}[f]^2.$$

and finally, since N is arbitrary,

$$\sum_{k>j} [a_k(k + \tfrac{1}{2})^{-\frac{1}{4}} - a_j(j + \tfrac{1}{2})^{-\frac{1}{4}}]^2 Q(k,j) < A\, \mathfrak{N}_{0,\alpha}[f]^2.$$

To establish the converse inequality let

$$f_r(x) = \sum_{0}^{r} a_k\omega_k(x).$$

We have

$$\mathfrak{N}_{0,\alpha}[f_r]^2 \simeq \sum_{0<j<k<r} [a_k(k + \tfrac{1}{2})^{-\frac{1}{4}} - a_j(j + \tfrac{1}{2})^{-\frac{1}{4}}]^2 Q(k,j) + \sum_{j=0}^{r} a_j^2(j + \tfrac{1}{2})^{-1} \sum_{k>r} Q(k,j).$$

Now

$$\sum_{0<j<k<r} [a_k(k + \tfrac{1}{2})^{-\frac{1}{4}} - a_j(j + \tfrac{1}{2})^{-\frac{1}{4}}]^2 Q(k,j) < \sum_{k>j} [a_k(k + \tfrac{1}{2})^{-\frac{1}{4}} - a_j(j + \tfrac{1}{2})^{-\frac{1}{4}}]^2 Q(k,j)$$

and

$$\sum_{j=0}^{r} a_j^2(j + \tfrac{1}{2})^{-1} \sum_{k>r} Q(k,j) < A \sum_{j=0}^{r} a_j^2(r - j + 1)^{-2\alpha}.$$

Applying Lemma 3c, we see that

$$\limsup_{r\to\infty} \mathfrak{N}_{0,\alpha}[f_r]^2 < A \sum_{k>j} [a_k(k + \tfrac{1}{2})^{-\frac{1}{4}} - a_j(j + \tfrac{1}{2})^{-\frac{1}{4}}]^2 Q(k,j).$$

The desired conclusion now follows from Lemma 4a.

This theorem is equivalent to the first relation announced at the end of §1. The second relation there can be obtained from the first on noting that $\mathfrak{N}_{\alpha,0}[f(x)] = \mathfrak{N}_{0,\alpha}[f(-x)]$.

5. Bounded multiplier transformations. We are now in a position to prove our main theorem in the case $\beta = 0$, $0 \leqslant \alpha < \frac{1}{2}$. Let S_μ be the multiplier transformation which carries

$$f(x) \sim \sum_0^\infty a_n \omega_n(x) \text{ into } S_N f(x) = \sum_0^N a_n \omega_n(x).$$

LEMMA 5a. *If* $0 \leqslant \alpha < \frac{1}{2}$ *then*

$$\mathfrak{N}_{\alpha,0}[S_N f] \leqslant A(\alpha)\mathfrak{N}_{\alpha,0}[f].$$

Proof. We may suppose $\alpha > 0$, since the case $\alpha = 0$ is trivial. By Theorem 4b we have

$$\begin{aligned}\mathfrak{N}_{0,\alpha}[S_N f]^2 \leqslant A \sum_{m,n<N} [a_n(n+\tfrac{1}{2})^{-\frac{1}{2}} - a_m(m+\tfrac{1}{2})^{-\frac{1}{2}}]^2 Q(m,n) \\ + A \sum_{\substack{m<N \\ n>N}} a_m^2 (m+\tfrac{1}{2})^{-1} Q(m,n).\end{aligned}$$

Using Theorem 4b again we see that

$$\sum_{m,n<N} [a_n(n+\tfrac{1}{2})^{-\frac{1}{2}} - a_m(m+\tfrac{1}{2})^{-\frac{1}{2}}]^2 Q(m,n) \leqslant A\mathfrak{N}_{0,\alpha}[f]^2.$$

Further

$$\begin{aligned}\sum_{\substack{m<N \\ n>N}} a_m^2 (m+\tfrac{1}{2})^{-1} Q(m,n) &\leqslant A \sum_{m<N} a_m^2 \sum_{n>N} (n-m)^{-1-2\alpha}, \\ &\leqslant A \sum_{m<N} a_m^2 (N+1-m)^{-2\alpha}, \\ &\leqslant A\, \mathfrak{N}_{0,\alpha}[f]^2,\end{aligned}$$

by Theorem 3b. These inequalities imply our desired result.

Let $b_\mu = 3.2^{\mu-2}$, $r_\mu = 2^{\mu-1}$, let σ_μ be the set of integers $b_\mu - r_\mu \leqslant k < b_\mu + r_\mu$, and let

$$\rho_\mu(x) = [1 - r_\mu^{-2}(x - b_\mu)^2].$$

If

$$f(x) \backsim \sum_0^\infty a_n \omega_n(x)$$

then we define

$$E_\mu(x) = \sum_{n \in \sigma_\mu} a_n \rho_\mu(n)\, \omega_n(x).$$

LEMMA 5b. *If* $0 \leqslant \alpha < \frac{1}{2}$ *then*

$$\sum_{\mu=0}^\infty \mathfrak{N}_{0,\alpha}[E_\mu]^2 \leqslant A\, \mathfrak{N}_{0,\alpha}[f]^2.$$

Proof. We may again suppose $\alpha > 0$. By Theorem 4b,

$$\mathfrak{N}_{0,\alpha}[E_\mu]^2 \simeq \sum\nolimits_1 + \sum\nolimits_2 + \sum\nolimits_3$$

where

$$\sum_1 = \sum_{\substack{m,n\in\sigma_\mu \\ n>m}} [\rho_\mu(n)\, a_n(n+\tfrac{1}{2})^{-\frac{1}{2}} - \rho_\mu(m)\, a_m(m+\tfrac{1}{2})^{-\frac{1}{2}}]^2 Q(n,m),$$

$$\sum_2 = \sum_{\substack{m\in\sigma_\mu \\ n>\sigma_\mu}} \rho_\mu(m)^2 a_m^2 (m+\tfrac{1}{2})^{-1} Q(n,m),$$

$$\sum_3 = \sum_{\substack{n\in\sigma_\mu \\ m<\sigma_\mu}} \rho_\mu(n)\, a_n^2 (n+\tfrac{1}{2})^{-1} Q(n,m).$$

Let us begin with $\sum_2$. For $m \in \sigma_\mu$ we have

$$\sum_{n>\sigma_\mu} Q(n,m) \leqslant Am(b_\mu + r_\mu - m)^{-2\alpha}$$

where A is indepdnent of m and μ. Since

$$\rho_\mu(m)^2 \leqslant 4(b_\mu + r_\mu - m)^2 r_\mu^{-2} \qquad (m \in \sigma_\mu)$$

it follows that

$$\sum_{n>\sigma_\mu} \rho_\mu(m)^2 (m+\tfrac{1}{2})^{-1} Q(n,m) \leqslant A(b_\mu + r_\mu - m)^{2-2\alpha} r_\mu^{-2} \qquad (m \in \sigma_\mu).$$

Making use of the inequalities

$$(b_\mu + r_\mu - m)^{2-2\alpha} \leqslant A(m+1)^{2-2\alpha} \qquad (m \in \sigma_\mu),$$

$$m + 1 \leqslant A\, r_\mu \qquad (m \in \sigma_\mu),$$

we obtain

$$\sum_{n>\sigma_\mu} \rho_\mu(m)^2 (m+\tfrac{1}{2})^{-1} Q(n,m) \leqslant A(m+1)^{-2\alpha} \qquad (m \in \sigma_\mu),$$

and thus

$$\sum_2 \leqslant A \sum_{m\in\sigma_\mu} |a_m|^2 (m+1)^{-2\alpha}.$$

We next turn to $\sum_3$, which follows the pattern established for $\sum_2$. For $n \in \sigma_\mu$ we have

$$\sum_{m<\sigma_\mu} Q(n,m) \leqslant A\, n(n - b_\mu + r_\mu)^{-2\alpha},$$

where A is independent of n and μ. Since

$$\rho_\mu(n)^2 \leqslant 4(n - b_\mu + r_\mu)^2 r_\mu^{-2}$$

it follows that

$$\sum_{m<\sigma_\mu} \rho_\mu(n)^2 (n+\tfrac{1}{2})^{-1} Q(n,m) \leqslant A(n - b_\mu + r_\mu)^{2-2\alpha} r_\mu^{-2}.$$

Making use of the inequalities

$$(n - b_\mu + r_\mu)^{2-2\alpha} \leqslant A(n+1)^{2-2\alpha} \qquad (n \in \sigma_\mu),$$

$$r_\mu \geqslant A(n+1) \qquad (n \in \sigma_\mu),$$

we obtain

$$\sum_{m<\sigma_\mu} \rho_\mu(n)^2(n+\tfrac{1}{2})\,Q(n,m) \leqslant A(n+1)^{-2\alpha} \qquad (n \in \sigma_\mu),$$

and thus

$$\sum_3 \leqslant A \sum_{n\in\sigma_\mu} |a_n|^2 (n+1)^{-2\alpha}.$$

It remains to treat $\sum_1$. Since

$$a_n\rho_\mu(n)(n+\tfrac{1}{2})^{-\frac{1}{2}} - a_m\rho_\mu(m)(m+\tfrac{1}{2})^{-\frac{1}{2}} = [a_n(n+\tfrac{1}{2})^{-\frac{1}{2}} - a_m(m+\tfrac{1}{2})^{-\frac{1}{2}}]\,\rho_\mu(n) + a_m(m+\tfrac{1}{2})^{-\frac{1}{2}}[\rho_\mu(n) - \rho_\mu(m)],$$

and since $0 \leqslant \rho_\mu(n) \leqslant 1$, we have

$$\sum_1 \leqslant 2 \sum_{\substack{n,m\in\sigma_\mu \\ n>m}} [a_n(n+\tfrac{1}{2})^{-\frac{1}{2}} - a_m(m+\tfrac{1}{2})^{-\frac{1}{2}}]^2 Q(n,m) + 2 \sum_{\substack{n,m\in\sigma_\mu \\ n>m}} a_m^2 (m+\tfrac{1}{2})^{-1}[\rho_\mu(n) - \rho_\mu(m)]^2 Q(n,m).$$

We assert that

$$\sum_{n=m+1}^{b_\mu+r_\mu} (m+\tfrac{1}{2})^{-1}[\rho_\mu(n) - \rho_\mu(m)]^2 Q(n,m) \leqslant A(m+1)^{-2\alpha} \qquad (m \in \sigma_\mu).$$

To verify this we note that

$$\rho_\mu(n) - \rho_\mu(m) = -(n-m)(n+m-2b_\mu)\,r_\mu^{-2},$$
$$|\rho_\mu(n) - \rho_\mu(m)| \leqslant A(n-m)\,r_\mu^{-1}.$$

It follows that

$$\sum_{n=m+1}^{b_\mu+r_\mu} (m+\tfrac{1}{2})^{-1}[\rho_\mu(n) - \rho_\mu(m)]^2 Q(n,m) \leqslant A r_\mu^{-2} \sum_{n=m+1}^{b_\mu+r_\mu} (n-m)^{1-2\alpha},$$
$$\leqslant A\,r_\mu^{-2}\,(b_\mu + r_\mu - m)^{2-2\alpha},$$
$$\leqslant A(m+1)^{-2\alpha},$$

as desired. Thus

$$\sum_1 \leqslant 2 \sum_{\substack{n,m\in\sigma_\mu \\ n>m}} [a_n(n+\tfrac{1}{2})^{-\frac{1}{2}} - a_m(m+\tfrac{1}{2})^{-\frac{1}{2}}]^2 Q(n,m) + A \sum_{n\in\sigma_\mu} a_n^2 (n+1)^{-2\alpha}.$$

Summing and applying Theorems 3b and 4b, we have

$$\sum_{\mu=2}^{\infty} \mathfrak{N}_{0,\alpha}[E_\mu]^2 \leqslant A\,\mathfrak{N}_{0,\alpha}[f]^2.$$

and our lemma is proved except for the fact that μ starts at 2 instead of 0, which is evidently without significance.

Let S_μ be the set of integers $2^{\mu-1} \leqslant k < 2^\mu$, $\mu = 1, 2, \ldots$; S_0 is the integer $k = 0$.

LEMMA 5c. *If* $0 < \alpha < \frac{1}{2}$ *and if* $n_\mu \in S_\mu$ *then*

$$\sum_{\mu=0}^{\infty} \sum_{m \in S_\mu} a_m^2 (|m - n_\mu| + 1)^{-2\alpha} \leqslant A(\alpha) \mathfrak{N}_{0,\alpha}[f]^2.$$

Proof. By Theorem 3b,

$$\sum_{m \in \sigma_\mu} \rho_\mu(m)^2 a_m^2 (|m - n_\mu| + 1)^{-2\alpha} \leqslant A\, \mathfrak{N}_{0,\alpha}[E_\mu]^2.$$

For $m \in S_\mu$ $\rho_\mu(m) \geqslant A$ and thus

$$\sum_{m \in S_\mu} a_m^2 (|m - n_\mu| + 1)^{-2\alpha} \leqslant A \sum_{m \in \sigma_\mu} \rho_\mu(m)^2 a_m^2 (|m - n_\mu| + 1)^{-2\alpha},$$
$$\leqslant A\, \mathfrak{N}_{0,\alpha}[E_\mu]^2.$$

Summing over μ and using Lemma 5b we obtain our desired result. Note that a similar inequality holds for $\mathfrak{N}_{\alpha,0}$.

THEOREM 5d. *If*

(i) $\quad f(x) \sim \sum_0^\infty a_n \omega_n(x) \qquad f \in \mathfrak{N}_{0,\alpha}, 0 \leqslant \alpha < \frac{1}{2},$

(ii) $\quad T = \{t_n\}_0^\infty$ belongs to class $\mathbf{M}(C)$,

(iii) $\quad Tf(x) \sim \sum_0^\infty t_n a_n \omega_n(x),$

then

$$\mathfrak{N}_{0,\alpha}[Tf] \leqslant AC\, \mathfrak{N}_{0,\alpha}[f].$$

Proof. We set

$$\delta_\mu(x) = \sum_{n \in S_\mu} a_n t_n \omega_n(x)$$

for $\mu = 0, 1, 2, \ldots$. If

$$F_M(x) = \sum_{\mu=0}^{M} \delta_\mu(x),$$

then, by Lemma 4a, it is enough to prove that

$$\mathfrak{N}_{0,\alpha}[F_M] \leqslant AC\, \mathfrak{N}_{0,\alpha}[f] \qquad (M = 1, 2, \ldots).$$

We have

$$\mathfrak{N}_{0,\alpha}[F_M]^2 \simeq \int_{-1}^{1} \Omega_\alpha(x) [F_M(x)]^2 dx;$$

and since

$$\int_{-1}^{1} \Omega_\alpha(x) [F_M(x)]^2 dx = \sum_{\mu=0}^{M} \int_{-1}^{1} \Omega_\alpha(x) [\delta_\mu(x)]^2 dx$$
$$+ \sum_{\mu, \nu=0, \mu \neq \nu}^{M} \int_{-1}^{1} \Omega_\alpha(x)\, \delta_\mu(x)\, \delta_\nu(x)\, dx,$$

it is sufficient to show that

$$5.1 \qquad \sum_{\mu,\nu=0,\mu\neq\nu}^{\infty} \left| \int_{-1}^{1} \Omega_\alpha(x)\, \delta_\mu(x)\, \delta_\nu(x)\, dx \right| \leqslant AC^2 \mathfrak{N}_{0,\alpha}[f]^2,$$

and

$$5.2 \qquad \sum_{\mu=0}^{\infty} \int_{-1}^{1} \Omega_\alpha(x)[\delta_\mu(x)]^2 dx \leqslant AC^2 \mathfrak{N}_{0,\alpha}[f]^2.$$

Let us set

$$I_{\nu,\mu} = \left| \int_{-1}^{1} \Omega_\alpha(x)\, \delta_\mu(x)\, \delta_\nu(x)\, dx \right| \qquad (\mu \neq \nu);$$

then

$$I_{\nu,\mu} = \left| \sum_{n\in S_\nu, m\in S_\mu} - a_n a_m t_n t_m (n+\tfrac{1}{2})^{-\frac{1}{2}} (m+\tfrac{1}{2})^{-\frac{1}{2}} c_{m,n} \right|,$$

and thus by Lemma 2b,

$$\begin{aligned} I_{\nu,\mu} &\leqslant AC^2 \sum_{n\in S_\nu, m\in S_\mu} |a_n a_m| (n+\tfrac{1}{2})^{-\frac{1}{2}} (m+\tfrac{1}{2})^{-\frac{1}{2}} Q(n,m), \\ &\leqslant AC^2 \sum_{n\in S_\nu, m\in S_\mu} [|a_n|^2 + |a_m|^2](n+\tfrac{1}{2})^{-\frac{1}{2}} (m+\tfrac{1}{2})^{-\frac{1}{2}} Q(n,m). \end{aligned}$$

From this we obtain

$$\sum_{\mu,\nu=0,\mu\neq\nu}^{\infty} I_{\mu,\nu} \leqslant AC^2 \sum_{\mu=0}^{\infty} \sum_{m\in S_\mu} |a_m|^2 \sum_{\substack{\nu=0\\ \nu\neq\mu}}^{\infty} \sum_{n\in S} (m+\tfrac{1}{2})^{-\frac{1}{2}} (n+\tfrac{1}{2})^{-\frac{1}{2}} Q(m,n).$$

Now

$$(n+\tfrac{1}{2})^{-\frac{1}{2}} (m+\tfrac{1}{2})^{-\frac{1}{2}} Q(n,m) \leqslant A|n-m|^{-1-2\alpha},$$

and if $m \in S_\mu$ then

$$\sum_{\nu=0,\,\nu\neq\mu}^{\infty} |n-m|^{-1-2\alpha} \leqslant A[(m-n_\mu'+1)^{-2\alpha} + (n_\mu''-m+1)^{-2\alpha}],$$

where n_μ' and n_μ'' are the first and last points of S_μ. Applying Lemma 5a it follows that

$$\begin{aligned} \sum_{\substack{\mu,\nu=0\\ \mu\neq\nu}}^{\infty} I_{\mu,\nu} &\leqslant AC^2 \sum_{\mu=0}^{\infty} \sum_{m\in S_\mu} |a_m|^2 [(|m-n_\mu'|+1)^{-2\alpha} + (|m-n_\mu''|+1)^{-2\alpha}], \\ &\leqslant AC^2 \mathfrak{N}_{0,\alpha}[f]^2, \end{aligned}$$

and 5.1 is seen to hold.

Let us next consider

$$\int_{-1}^{1} \Omega_\alpha(x)[\delta_\mu(x)]^2 dx.$$

For μ fixed we set

$$s(n,x) = \sum_{k_\mu-r_\mu}^{n} a_m \rho_\mu(m)\, \omega_m(x).$$

It follows from Lemma 5a that

$$\mathfrak{N}_{0,\alpha}[s(n,x)] \leqslant A\mathfrak{N}_{0,\alpha}[E_\mu]. \tag{5.3}$$

We have, if $u(n) = t_n/\rho_\mu(n)$,

$$\delta_\mu(x) = \sum_{n\in S_\mu} u(n)[s(n,x) - s(n-1,x)].$$

Summing by parts we find that

$$\delta_\mu(x) = \sum_{n\in S_\mu} s(n,x)\,[u(n) - u(n+1)] + u(2^\mu)\,s(2^\mu - 1, x) - u(2^{\mu-1})\,s(2^{\mu-1} - 1, x),$$

from which using 5.3 it follows that

$$\mathfrak{N}_{0,\alpha}[\delta_\mu] \leqslant A\,\mathfrak{N}_{0,\alpha}[E_\mu]\{\sum_{n\in S_\mu} |u(n) - u(n+1)| + |u(2^\mu)| + |u(2^{\mu-1})|\}.$$

Now it is easily verified that

$$\sum_{n\in S_\mu} |u(n) - u(n+1)| + |u(2^\mu)| + |u(2^{\mu-1})| \leqslant AC$$

and thus

$$\mathfrak{N}_{0,\alpha}[\delta_\mu] \leqslant AC\mathfrak{N}_{0,\alpha}[E_\mu].$$

Squaring and summing over μ we see using Lemma 5b that 5.2 holds.

6. Multiplier transformations, continued. Let $P(\beta, \alpha)$ stand for the proposition that if $T \in \mathbf{M}(C)$ then $\mathfrak{N}_{\beta,\alpha}[Tf] \leqslant AC\,\mathfrak{N}_{\beta,\alpha}[f]$ where A depends only on α and β. Theorem 5d shows that $P(0, \alpha)$ is valid if $0 \leqslant \alpha < \frac{1}{2}$. We wish to show that $P(\beta, \alpha)$ is valid for $(-\frac{1}{2} < \beta, \alpha < \frac{1}{2})$. We begin with two general principles.

LEMMA 6a. *If $P(\beta, \alpha)$ is valid so is $P(\alpha, \beta)$.*

Proof. Let $f(x) \in \mathfrak{N}_{\alpha,\beta}$; then, if $F(x) = f(-x)$,

$$\mathfrak{N}_{\alpha,\beta}[f] = \mathfrak{N}_{\beta,\alpha}[F], \qquad \mathfrak{N}_{\alpha,\beta}[Tf] = \mathfrak{N}_{\beta,\alpha}[TF].$$

LEMMA 6b. *If $P(\beta, \alpha)$ is valid so is $P(-\beta, -\alpha)$.*

Proof. Let $f(x) \in \mathfrak{N}_{\beta,\alpha}$. Let

$$g(x) = \sum_0^r b_k\omega_k(x).$$

We have

$$\left| \int_{-1}^1 [Tf(x)]\,g(x)\,dx \right| \leqslant \mathfrak{N}_{\beta,\alpha}[Tf]\,\mathfrak{N}_{-\beta,-\alpha}[g] \leqslant AC\mathfrak{N}_{\beta,\alpha}[f]\,\mathfrak{N}_{-\beta,-\alpha}[g],$$

and since

$$\int_{-1}^1 [Tf(x)]\,g(x)\,dx = \int_{-1}^1 f(x)[Tg(x)]\,dx,$$

this can be rewritten as

$$\left| \int_{-1}^{1} f(x)[Tg(x)]\,dx \right| \leqslant CA\ \mathfrak{N}_{\beta,\alpha}[f]\ \mathfrak{N}_{-\beta,-\alpha}[g],$$

which, since it is true for every $f \in \mathfrak{N}_{\beta,\alpha}$, implies that $\mathfrak{N}_{-\beta,-\alpha}[Tg] \leqslant CA\ \mathfrak{N}_{-\beta,-\alpha}[g]$. An evident approximation argument enables us to remove the restriction that $g(x)$ be a finite sum of Legendre polynomials.

Lemmas 6a and 6b together imply that $P(0,\alpha)$ and $P(\beta,0)$ are true for $-\frac{1}{2} < \alpha, \beta < \frac{1}{2}$.

LEMMA 6c. *$P(\beta,\alpha)$ is valid if $-\frac{1}{2} < \alpha, \beta \leqslant 0$.*

Proof. Let $f \in \mathfrak{N}_{\beta,\alpha}$. We have

$$\begin{aligned}\mathfrak{N}_{\beta,\alpha}[Tf] &\leqslant A\ \mathfrak{N}_{0,\alpha}[Tf] + A\ \mathfrak{N}_{\beta,0}[Tf],\\ &\leqslant CA\ \mathfrak{N}_{0,\alpha}[f] + CA\ \mathfrak{N}_{\beta,0}[f],\end{aligned}$$

where we have used the fact that $P(0,\alpha)$, $P(\beta,0)$ are true for $-\frac{1}{2} < \alpha, \beta \leqslant 0$. Since

$$\mathfrak{N}_{0,\alpha}[f] \leqslant A\ \mathfrak{N}_{\beta,\alpha}[f], \qquad \mathfrak{N}_{\beta,0}[f] \leqslant A\ \mathfrak{N}_{\beta,\alpha}[f],$$

we have

$$\mathfrak{N}_{\beta,\alpha}[Tf] \leqslant CA\ \mathfrak{N}_{\beta,\alpha}[f],$$

as desired.

Lemma 6c in conjunction with our previous results shows that $P(\beta,\alpha)$ is valid if $-\frac{1}{2} < \alpha, \beta < \frac{1}{2}$, and if $\alpha\beta > 0$; that is if α and β are of the same sign. The case where α and β are of different signs is slightly more difficult.

LEMMA 6d. *If $0 \leqslant \alpha, \beta < \frac{1}{2}$, if $T \in \mathbf{M}(C)$, and if*

$$F(x) = (1-x)\,T[f(x)] - T[(1-x)\,f(x)]$$

then

$$\mathfrak{N}_{-\beta,0}[F] \leqslant AC\ \mathfrak{N}_{0,\alpha}[f]$$

where A depends only on α and β.

Proof. It is enough to prove this in the case where $f(x)$ is a finite sum of Legendre polynomials. We have

$$f(x) = \sum_{0}^{\infty} a_k \omega_k(x) = \sum_{0}^{\infty} R_k P_k(x), \qquad R_k = a_k(k+\tfrac{1}{2})^{\frac{1}{2}}.$$

Since (9, p. 308),

$$(1-x)\,P_n(x) = -(n+1)(2n+1)^{-1}P_{n+1}(x) + P_n(x) - n(2n+1)^{-1}P_{n-1}(x),$$

we find, after a short computation, that $F(x) = F_1(x) + F_2(x)$ where

$$F_1(x) = \sum_{0}^{\infty} (n+1)(2n+1)^{-1}R_n[-t_n + t_{n+1}]\,P_{n+1}(x),$$

$$F_2(x) = \sum_{0}^{\infty} n(2n+1)^{-1}R_n[-t_n + t_{n-1}]\,P_{n-1}(x).$$

Note that these series are only formally infinite. Let $g(x) \in \mathfrak{N}_{\beta,0}$ and let

$$g(x) \sim \sum_0^\infty b_k \omega_k(x) = \sum_0^\infty R_k' P_k(x) R_k' = b_k (k + \tfrac{1}{2})^{\frac{1}{2}}.$$

We have

$$\int_{-1}^1 F_1(x)\, g(x)\, dx = 2 \sum_0^\infty (n+1)(2n+1)^{-1}(2n+1)^{-1} R_n R_{n+1}' [-t_n + t_{n+1}],$$

$$\left| \int_{-1}^1 F(x)\, g(x)\, dx \right| \leqslant A \sum_{\mu=0}^\infty \sum_{n \in S_\mu} |a_n b_{n+1}| |t_n - t_{n+1}|.$$

If $\alpha_\mu = \text{l.u.b.} |a_n|$ for $n \in S_\mu$ and $\beta_\mu = \text{l.u.b.} |b_n|$ for $n \in S_\mu$ then (see the remark following the proof of Lemma 5c)

$$\begin{aligned} \left| \int_{-1}^1 F_1(x)\, g(x)\, dx \right| &\leqslant A \sum_{\mu=0} \alpha_\mu (\beta_\mu + \beta_{\mu+1}) \sum_{n \in S_\mu} |t_n - t_{n+1}| \\ &\leqslant AC \sum_{\mu=0}^\infty \alpha_\mu (\beta_\mu + \beta_{\mu+1}) \leqslant AC \Big(\sum_{\mu=0}^\infty \alpha_\mu^2 \Big)^{\frac{1}{2}} \Big(\sum_{\mu=0}^\infty \beta_\mu^2 \Big)^{\frac{1}{2}} \\ &\leqslant AC\, \mathfrak{N}_{0,\alpha}[f]\, \mathfrak{N}_{\beta,0}[g]. \end{aligned}$$

Since this is true for every $g \in \mathfrak{N}_{\beta,0}$ it implies that $\mathfrak{N}_{-\beta,0}[F_1] \leqslant AC\, \mathfrak{N}_{0,\alpha}[f]$. Similarly we can show that $\mathfrak{N}_{-\beta,0}[F_2] \leqslant AC\, \mathfrak{N}_{0,\alpha}[f]$.

LEMMA 6e. *$P(\beta, \alpha)$ is valid if $-\frac{1}{2} < \beta \leqslant 0 \leqslant \alpha < \frac{1}{2}$.*

Proof. We have

$$\mathfrak{N}_{\beta,\alpha}[Tf] \leqslant A\, \mathfrak{N}_{0,\alpha}[Tf] + A\, \mathfrak{N}_{\beta,0}[(1-x)\, Tf].$$

Since $P(0, \alpha)$ is valid

$$\mathfrak{N}_{0,\alpha}[Tf] \leqslant AC\, \mathfrak{N}_{0,\alpha}[f].$$

If $F(x)$ is defined as in Lemma 6d, then

$$\begin{aligned} \mathfrak{N}_{\beta,0}[(1-x)\, Tf] &= \mathfrak{N}_{\beta,0}[T\{(1-x)\, f(x)\} + F(x)] \\ &\leqslant \mathfrak{N}_{\beta,0}[T\{(1-x)\, f(x)\}] + \mathfrak{N}_{\beta,0}[F(x)]. \end{aligned}$$

By $P(\beta, 0)$

$$\begin{aligned} \mathfrak{N}_{\beta,0}[T\{(1-x)\, f(x)\}] &\leqslant AC\, \mathfrak{N}_{\beta,0}[(1-x)\, f(x)] \\ &\leqslant AC\, \mathfrak{N}_{\beta,\alpha}[f(x)]. \end{aligned}$$

Since $f \in \mathfrak{N}_{0,\alpha}$, Lemma 6d implies that

$$\mathfrak{N}_{\beta,0}[F(x)] \leqslant AC\, \mathfrak{N}_{0,\alpha}[f] \leqslant AC\, \mathfrak{N}_{\beta,\alpha}[f].$$

Thus combining these results

$$\mathfrak{N}_{\beta,\alpha}[Tf] \leqslant AC\, \mathfrak{N}_{\beta,\alpha}[f],$$

as desired.

Our lemmas yield

THEOREM 6f. *$P(\alpha, \beta)$ is valid for $-\frac{1}{2} < \alpha, \beta < \frac{1}{2}$.*

It is to be noted that there is another general principle which we have not invoked. It is easily shown by arguments like those used in the Riesz-Thorin convexity theorem that if $P(\alpha', \beta')$ and $P(\alpha'', \beta'')$ are valid then so is $P(\alpha, \beta)$ where $\alpha = \alpha'(1-\theta) + \alpha''\theta$, $\beta = \beta'(1-\theta) + \beta''\theta$, $0 \leqslant \theta \leqslant 1$. However this would not have shortened our arguments.

We shall now indicate briefly and without going into detail how it can be seen that the restriction $-\frac{1}{2} < \alpha, \beta < \frac{1}{2}$ is essential in Theorem 6f. If Theorem 6f were valid for α and β not satisfying the above restriction then 1.2 of §1 would hold. That this is impossible can be seen using the methods of Newman and Rudin **(5)**.

7. Fractional integration. Let $f(x) \in \mathfrak{N}_{\alpha,\beta}$ where $-\frac{1}{2} < \alpha, \beta < \frac{1}{2}$ and let $f(x)$ have mean value zero so that

$$f(x) \sim \sum_1^\infty a_n \omega_n(x).$$

We set

$$I(\sigma) f(x) \sim \sum_1^\infty [n(n+1)]^{-\sigma/2} a_n \omega_n(x).$$

$I(\sigma)$ is a fractional integration operator if σ is positive and a fractional differentiation operator if σ is negative. Note that

$$I(-2) f(x) = [(x^2 - 1) f'(x)]'.$$

In the present section we shall apply our results to the study of $I(\sigma)$.

THEOREM 7a. *Let $f(x)$ have mean value zero. If*

(i) $-\frac{1}{2} < \alpha_1, \beta_1 < \frac{1}{2}$, $-\frac{1}{2} < \alpha_2, \beta_2 < \frac{1}{2}$,
(ii) $0 < \theta < 1$,
(iii) $\sigma = \sigma_2\theta$, $\alpha = (1-\theta)\alpha_1 + \theta\alpha_2$, $\beta = (1-\theta)\beta_1 + \theta\beta_2$,

then

$$\mathfrak{N}_{\alpha,\beta}[I(\sigma) f] \leqslant A\, \mathfrak{N}_{\alpha_1,\beta_1}[f]^{1-\theta}\, \mathfrak{N}_{\alpha_2,\beta_2}[I(\sigma_2) f]^{\theta}$$

where A depends only upon $\alpha_1, \beta_1, \alpha_2, \beta_2, \sigma_2$ and θ, but not upon f.

Using Theorem 6f we can show that, if $f(x)$ has mean value zero and if $-\frac{1}{2} < \alpha, \beta < \frac{1}{2}$, then

$$\mathfrak{N}_{\alpha,\beta}[I(i\tau) f] \leqslant A(\tau)\, \mathfrak{N}_{\alpha,\beta}[f]$$

where $A(\tau) = 0(|\tau|)$ as $\tau \to \pm\infty$. Note that although Theorem 6f is stated for real multipliers, it evidently holds for complex multipliers as well. This can be seen by decomposing into real and imaginary parts. The inequality we have just established asserts that "fractional integration" of purely imaginary order is a bounded transformation of $\mathfrak{N}_{\alpha,\beta}$ into itself. It now follows from a general theorem on groups of multiplier transformations that this property of fractional integration of purely imaginary order implies **(3; 11)** the conclusion of Theorem 7a.

We shall now use Theorem 7a to prove an analogue of a classical theorem of Hardy and Littlewood (**10**, p. 227). The method employed here could also be used to give a new demonstration of their result.

THEOREM 7b. *Let $f(x)$ have mean value zero. If*

(i) $\alpha_1 = \alpha - \sigma, \beta_1 = \beta - \sigma$ $(\sigma > 0)$,

(ii) $\alpha, \beta < \frac{1}{2}, -\frac{1}{2} < \alpha_1, \beta_1$,

then

$$\mathfrak{N}_{\beta_1,\alpha_1}[I(\sigma) f] \leqslant A\, \mathfrak{N}_{\beta,\alpha}[f],$$

where A depends only upon α, β and σ but not upon f.

Proof. It is enough to prove this for $f(x)$ a finite sum of Legendre polynomials. We first assert that if $-\frac{1}{2} < \alpha < \frac{1}{2}$ then

$$\mathfrak{N}_{0,\alpha_1}[I(\sigma) f] \leqslant A\, \mathfrak{N}_{0,\alpha}[f], \tag{7.1}$$

where $\sigma > 0$, $\alpha_1 = \alpha - \sigma$, $\alpha_1 > -\frac{1}{2}$. Suppose first that $\alpha_1 \leqslant 0 \leqslant \alpha$. Let $\mathfrak{N}_{0,-\alpha_1}[g] \leqslant 1$, where

$$g(x) \sim \sum_0^\infty b_n \omega_n(x).$$

We have

$$\begin{aligned}
\left| \int_{-1}^1 [I(\sigma) f(x)]\, g(x)\, dx \right| &= \left| \sum_1^\infty a_n [n(n+1)]^{-\frac{1}{2}\sigma} b_n \right| \\
&\leqslant A \left[\sum_1^\infty a_n^2 (n+1)^{-2\alpha} \right]^{\frac{1}{2}} \left[\sum_1^\infty b_n^2 (n+1)^{+2\alpha_1} \right]^{\frac{1}{2}} \\
&\leqslant A\, \mathfrak{N}_{0,\alpha}[f]\, \mathfrak{N}_{0,-\alpha_1}[g] \\
&\leqslant A\, \mathfrak{N}_{0,\alpha}[f].
\end{aligned}$$

Here we have used Theorem 3b. Since this is true for an arbitrary g such that $\mathfrak{N}_{0,-\alpha_1}[g] \leqslant 1$, it implies 7.1 for $-\frac{1}{2} < \alpha_1 \leqslant 0 \leqslant \alpha < \frac{1}{2}$.

Next assume that $0 \leqslant \alpha_1 < \alpha < \frac{1}{2}$, where $\alpha_1 = \alpha - \sigma$. We have from the case already considered,

$$\mathfrak{N}_{0,0}[I(\alpha) f] \leqslant A\, \mathfrak{N}_{0,\alpha}[f].$$

By Theorem 7a, if $\sigma = \theta\alpha$

$$\mathfrak{N}_{0,\alpha_1}[I(\sigma) f] \leqslant A\, \mathfrak{N}_{0,\alpha}[f]^{1-\theta} \mathfrak{N}_{0,0}[I(\alpha) f]^{\theta} \leqslant A\, \mathfrak{N}_{0,\alpha}[f].$$

Thus 7.1 is true if $0 \leqslant \alpha_1 < \alpha < \frac{1}{2}$. If $-\frac{1}{2} < \alpha_1 < \alpha \leqslant 0$, let $\mathfrak{N}_{0,-\alpha_1}[g] \leqslant 1$. We have

$$\left| \int_{-1}^1 [I(\sigma) f(x)]\, g(x)\, dx \right| = \left| \int_{-1}^1 f(x) [I(\sigma)\, g(x)]\, dx \right| \leqslant \mathfrak{N}_{0,\alpha}[f]\, \mathfrak{N}_{0,-\alpha}[I(\sigma)\, g(x)].$$

But

$$\mathfrak{N}_{0,-\alpha}[I(\sigma)\, g(x)] \leqslant A\, \mathfrak{N}_{0,-\alpha_1}[g(x)] \leqslant A$$

(because $-\alpha = -\alpha_1 - \sigma$, $0 \leqslant -\alpha < -\alpha_1 < \frac{1}{2}$), so that

$$\left| \int_{-1}^{1} [I(\sigma)\, f(x)]\, g(x)\, dx \right| \leqslant A\, \mathfrak{N}_{0,\alpha}[f].$$

Since this is true for every g with $\mathfrak{N}_{0,-\alpha_1}[g] \leqslant 1$, it implies 7.1 for $-\frac{1}{2} < \alpha_1 < \alpha \leqslant 0$. Thus, finally, 7.1 is true if $-\frac{1}{2} < \alpha_1 < \alpha < \frac{1}{2}$. Similarly we can show that if $-\frac{1}{2} < \beta_1 < \beta < \frac{1}{2}$, $\beta_1 = \beta - \sigma$ $(\sigma > 0)$, then

7.2 $$\mathfrak{N}_{\beta_1,0}[I(\sigma)\, f] \leqslant A\, \mathfrak{N}_{\beta,0}[f].$$

Let us set

$$f_1(x) = \tfrac{1}{2}[1 - \operatorname{sgn} x]\, f(x),$$
$$f_2(x) = \tfrac{1}{2}[1 + \operatorname{sgn} x]\, f(x).$$

We have from 7.2 that

7.3 $$\mathfrak{N}_{\beta_1,0}[I(\sigma)\, f_1] \leqslant A\, \mathfrak{N}_{\beta,0}(f_1) \leqslant A\, \mathfrak{N}_{\beta,\alpha}[f].$$

Since $I(\sigma) \in \mathbf{M}(C)$ (for some C) Theorem 6f gives

7.4 $$\mathfrak{N}_{\beta,\alpha_1}[I(\sigma)\, f_1] \leqslant A\, \mathfrak{N}_{\beta,\alpha_1}[f_1] \leqslant A\, \mathfrak{N}_{\beta,\alpha}[f].$$

The inequalities (3) and (4) together yield

7.5 $$\mathfrak{N}_{\beta_1,\alpha_1}[I(\sigma)\, f_1] \leqslant A\, \mathfrak{N}_{\beta_1,0}[I(\sigma)\, f_1] + A\, \mathfrak{N}_{\beta,\alpha_1}[I(\sigma)\, f_1] \leqslant A\, \mathfrak{N}_{\beta,\alpha}[f].$$

Similarly from 7.1 we have that

7.6 $$\mathfrak{N}_{0,\alpha_1}[I(\sigma)\, f_2] \leqslant A\, \mathfrak{N}_{0,\alpha}[f_2] \leqslant A\, \mathfrak{N}_{\beta,\alpha}[f].$$

Again, since $I(\sigma) \in \mathbf{M}(C)$ (for some C),

7.7 $$\mathfrak{N}_{\beta_1,\alpha}[I(\sigma)\, f_2] \leqslant A\, \mathfrak{N}_{\beta_1,\alpha}[f_2] \leqslant A\, \mathfrak{N}_{\beta,\alpha}[f],$$

and hence

7.8 $$\mathfrak{N}_{\beta_1,\alpha_1}[I(\sigma)\, f_2] \leqslant A\, \mathfrak{N}_{0,\alpha_1}[I(\sigma)\, f_2] + A\, \mathfrak{N}_{\beta_1,\alpha}[I(\sigma)\, f_2] \leqslant A\, \mathfrak{N}_{\beta,\alpha}[f].$$

Making use of 7.5 and 7.8, we obtain

$$\mathfrak{N}_{\beta_1,\alpha_1}[I(\sigma)\, f] \leqslant \mathfrak{N}_{\beta_1,\alpha_1}[I(\sigma)\, f_1] + \mathfrak{N}_{\beta_1,\alpha_1}[I(\sigma)\, f_2] \leqslant A\, \mathfrak{N}_{\beta,\alpha}[f],$$

as desired.

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METAMATHEMATICAL CONSIDERATIONS ON THE RELATIVE IRREDUCIBILITY OF POLYNOMIALS

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1. Introduction. In this paper, infinite fields K will be discussed which satisfy the following condition:

CONDITION C: for any polynomial $p(t, x)$ in x, coefficients in $K(t)$, t transcendental with respect to K, which has no zeros in $K(t)$, there is a t^* in K for which $p(t^*, x)$ has no zeros in K.

This condition is *a priori* weaker than Franz's necessary and sufficient criterion[1] in the field K. As an illustration of the scope of the metamathematical method of proof in algebra, a metamathematical theorem (2.1) will be established for fields satisfying condition C. By means of the theorem it will be shown that for fields satisfying condition C not only does Hilbert's theorem hold, but also two other theorems, one related to results proven by Dörge **(1)** and the other new. The metamathematical theorem to be established is an extension of a theorem proven by Robinson **(6**, pp. 35–52**)**.

Throughout this paper K will denote an infinite field, t an element transcendental over K, and x will be an indeterminate. We will use the usual notation of square and round brackets when adjoining elements to a field to denote the ring of the polynomials in the adjoined elements and the field of rational functions of the adjoined elements respectively. We will further use for logical notation, variables x, y, z, u, v, x' and y', "$\sim$," "$\wedge$," "$\vee$," and "$\supset$" for negation, conjunction, disjunction and implication respectively, and "(x)" and "$(\exists x)$" for universal and existential quantification respectively.

2. A metamathematical theorem. The language of the first order predicate calculus is assumed given with any number of individual parameters and atomic predicates. For any given field K and any element t transcendental with respect to K, the language $\mathfrak{F}$ is the language of the first order predicate calculus applied in the following way: to each member of K and to t is assigned an individual parameter, and these are the individual parameters of $\mathfrak{F}$; to each subset of K, to each subset of the set $K \times K$ of pairs from K, to each subset of the set $K \times K \times K$ of triples from K, etc., is assigned an atomic predicate, and these are the atomic predicates of $\mathfrak{F}$. Finally the language $\mathfrak{L}$ is defined to be identical with $\mathfrak{F}$ except that it is lacking an individual parameter for t.

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[1]See Franz **(2**, §3**)**. In his paper, Franz considers fields K of the form $K = F(\alpha)$, where F is an infinite field and α is transcendental over F. For finite F, see Inaba **(4)**.

This assignment of parameters and predicates for the languages $\mathfrak{F}$ and $\mathfrak{L}$ will remain fixed for this paper, so that when we speak of a statement of $\mathfrak{L}$ or $\mathfrak{F}$ holding for K or $K(t)$, it is assumed to hold under the given assignment. Then $\mathfrak{K}$ is defined to be the set of all statements of $\mathfrak{L}$ holding for $\mathfrak{K}$; i.e., $\mathfrak{K}$ is the largest set of statements of $\mathfrak{L}$ for which K is a model.

THEOREM 2.1. *For any field K fulfilling condition C, there is an extension S' of $S = K(t)$ which is a model of $\mathfrak{K}$ and for which every member of $S' - S$ is transcendental with respect to S.*

Proof. Let the atomic predicates $E(x, y)$, $S(x, y, z)$ and $P(x, y, z)$ be those of $\mathfrak{F}$ which have been assigned to the following sets respectively: the set of all pairs of members of K with the first member of the pair equal to the second member of the pair, the set of all triples from K with the sum of the first two members equal to the third member, and the set of all triples from K with the product of the first two members equal to the third member. Within $\mathfrak{F}$ can then be expressed any polynomial equation $p(t) = 0$ by using the atomic predicates E, S and P, and by using the individual parameters in F corresponding to the coefficients of the polynomials $p(t)$. Thus using the letter τ as the individual parameter of $\mathfrak{F}$ which has been assigned to the element t transcendental with respect to K, the notation

$$\sum_{i=0}^{m} p_i \tau^i \neq 0$$

can be used as an abbreviation for the full statement in $\mathfrak{F}$ for the polynomial equation of the same form. The following sets of statements from $\mathfrak{F}$ are then defined:

$\mathfrak{S}_1$: the set of all statements of the form

$$\sum_{i=0}^{m} p_i \tau^i \neq 0$$

for all m, and for all sets of individual parameters for which the polynomial $p(t) \in K[t]$ with coefficients corresponding to the parameters p_i, is not null.

$\mathfrak{S}_2$: the set of all statements of the form

$$(x)\left(\sum_{i=0}^{m}\sum_{j=0}^{n} q_{ij} \tau^i x^j \neq 0\right)$$

for all m and n, and for all sets of individual parameters for which the polynomial $q(t, x) \in K[t, x]$ with coefficients corresponding to the parameters q_{ij} is irreducible in x over $K(t)$, and of degree greater than one in x.

The set $\mathfrak{K} \cup \mathfrak{S}_1 \cup \mathfrak{S}_2$ of statements from $\mathfrak{F}$ is consistent, provided we can show that K is a model for any finite subset of the set. Now, given any finite subset of $\mathfrak{K} \cup \mathfrak{S}_1 \cup \mathfrak{S}_2$, let $p_i(t)$ for $i = 1, \ldots, \mu$ be the polynomials in $K[t]$ used to define the statements from $\mathfrak{S}_1$, and let $q_j(t, x)$ for $j = 1, \ldots, \nu$ be the polynomials in $K[t, x]$ used to define the statements from S_2. Then

to show that K is a model for this finite subset of statements, we need only find a suitable $t^* \in K$ to which the parameter τ can be assigned, since the given assignment of parameters and predicates is assumed to be fixed. But then t^* need only be chosen so that $p(t^*, x)$ has no root x in K, where the polynomial $p(t, x)$ is defined to be

$$\prod_{i=1}^{n} p_i(t) \prod_{j=1}^{r} q_j(t, x).$$

Since all the q_j are irreducible in x, $p(t, x)$ has no roots in $K(t)$, and therefore by condition C there is a $t^* \in K$ for which $p(t^*, x)$ has no root in K.

Since the set $\mathfrak{R} \cup \mathfrak{S}_1 \cup \mathfrak{S}_2$ is consistent, it has a model[3] M. Within $\mathfrak{R}$ will be a set of statements containing no quantifiers or variables but only individual parameters of $\mathfrak{L}$, and expressing all possible algebraic relationships between members of K. All of these statements are valid in the model M so that M must contain a subset K' which is a field isomorphic to K. Further, since M is a model for the set $\mathfrak{S}_1$, M must contain a member t' which is transcendental with respect to K' and therefore must contain a subset $K'(t')$ isomorphic with $K(t)$. Since M is also a model of $\mathfrak{S}_2$, every member of $M - K'(t')$ is transcendental with respect to $K'(t')$. Therefore an extension S' of S can be constructed isomorphic to M and such that every member of $S' - S$ is transcendental with respect to S, establishing the theorem.

3. Applications. Theorem 2.1 will now be applied to establish several results for fields fulfilling condition C. The principle of our method is to establish that a certain statement of $\mathfrak{L}$ holds in S'. It then follows from 2.1 that the statement holds also in K (see proof of Theorem 3.1), although it may be far more difficult to prove this fact directly.

THEOREM 3.1. *If K fulfils condition C, then for any irreducible polynomial $p(t, x)$ in x with coefficients in $K(t)$ there are infinitely many t^* from K such that $p(t^*, x)$ is irreducible in x over K.*

Proof. Let S and S' be given for K as in Theorem 2.1. We will first show that if a polynomial $p(t, x)$ with coefficients in S is irreducible in S then it is also irreducible in S'. For if $p = p_1 \cdot p_2$, with $p_1, p_2 \in S'[x]$, is a non-trivial factorization of p, then in the algebraic closure T' of S', p_1 and p_2 will split into linear factors. But in the algebraic closure T of S, p will split into linear factors and these factors can be identified with factors of p_1 and p_2 in T'. Hence p_1 and p_2 must split into linear factors in T and must therefore be contained in $T[x]$. But since all members of T are algebraic with respect to S, $T \cap S' = S$, and hence $p_1, p_2 \in S[x]$, giving that p is reducible in S.

For any polynomial $p(t, x)$ in x with coefficients from $K(t)$ there can be defined a predicate $I(z)$ in the language $\mathfrak{L}$ such that for any member k of K,

[3]Compare Robinson (5) or Henkin (3).

$I(k)$ expresses that $p(k, x)$ is irreducible in x over K. We can assume that $p(t, x)$ is a polynomial

$$\sum_{i=0}^{n} f_i(t)\, x^i,$$

where $f_i(t) \in K[t]$ for $0 \leqslant i \leqslant n$, since a factor $q(t) \in K[t]$ does not affect the irreducibility of $p(t, x)$. Then for any given $k \in K$, the fact that $p(k, x)$ has no factor of degree m in x can be expressed by $I_m(k)$, which is defined to be:

$$(x_0) \ldots (x_m)(y_0) \ldots (y_{n-m})(x_0\, y_0 \neq f_0(k) \vee x_0\, y_1 + x_1\, y_0 \neq f_1(k) \vee \ldots \vee x_m\, y_{n-m} \neq f_n(k))$$

where, for example, $x_0\, y_0 \neq f_0(k)$ is an abbreviation for

$$(z)\left(P(x_0, y_0, z) \supset z \neq \sum_{i=0}^{r} p_i\, k^i\right)$$

assuming $f_0(t)$ to be the polynomial

$$\sum_{i=0}^{r} p_i\, t^i$$

and where the other members of the disjunction are abbreviations for corresponding statements. Then if $I(k)$ is defined to be $I_1(k) \wedge I_2(k) \wedge \ldots \wedge I_{n-1}(k)$, $I(k)$ expresses that $p(k, x)$ for $k \in K$ is irreducible in x over K. Hence to show that there is a t^* in K for which $p(t^*, x)$ is irreducible over K, we need only to show that $(\exists x) I(x)$ is valid in K. Further, in order to show that there are infinitely many t^* in K for which $p(t^*, x)$ is irreducible over K, we need only prove that

$$\text{(I)} \quad (\exists z_1)(\exists z_2) \ldots (\exists z_n)(z_1 \neq z_2 \wedge z_1 \neq z_3 \wedge z_2 \neq z_3 \wedge \ldots \wedge z_{n-1} \neq z_n \wedge I(z_1) \wedge I(z_2) \wedge \ldots \wedge I(z_n))$$

is valid in K for any n.

To show that a statement is valid for K it is only necessary to show that it is valid for S', since if a statement is valid for S' but not valid for K, its negation would be valid for K and therefore a member of $\mathfrak{R}$, contradicting that S' is a model for $\mathfrak{R}$. But $(\exists x) I(x)$ is valid for S' since t is a member of S', and therefore since $p(t, x)$ is irreducible over S it must also be irreducible over S'. Also the statement (I) is valid for S' since not only $p(t, x)$ is irreducible over S but also $p(t+k, x)$ for any member k of K. Thus the theorem is established.

This theorem shows that for any field fulfilling condition C, Hilbert's irreducibility theorem can be proven.

In **(1)**, Dörge has proven theorems related to Hilbert's irreducibility theorem as the following theorem is related to Theorem 3.1.

THEOREM 3.2. *If K is an ordered field fulfilling condition C, then for any a and b from K for which $a < b$ and for any irreducible polynomial $p(t, x)$ in x with coefficients in $K(t)$, there exists a t^* in K such that $a < t^* < b$ and such that $p(t^*, x)$ is irreducible in x over K.*

Proof. The order in K defines a set of pairs of K to which there will be assigned a predicate $Q(x, y)$ of $\mathfrak{L}$. Q will appear together with E, S and P in statements in $\mathfrak{K}$ expressing that K is an ordered field. For a given irreducible polynomial $p(t, x)$, $I(z)$ will be defined as in the proof of Theorem 3.1. Then for the present theorem it is sufficient to prove that for any parameters α and β of $\mathfrak{L}$

$$(\text{II}) \qquad (\exists z)(Q(\alpha, \beta) \supset Q(\alpha, z) \wedge Q(z, \beta) \wedge I(z))$$

is valid for S' (and hence that it is valid for K, by Theorem 2.1).

Because S' is a model of $\mathfrak{K}$, it must be an ordered field under some ordering $\prec$ to which Q is assigned. To show then that (II) is valid for S', it is sufficient to show that for any a and b in K such that $a \prec b$, there is a w in S' which satisfies $I(z)$ and is such that $a \prec w \prec b$. But such a w can always be found, for if $1 \prec t$, put $w = a + (b - a)t^{-1}$; if $0 \prec t \prec 1$, put $w = a + (b - a)t$; if $-1 \prec t \prec 0$, put $w = a + (a - b)t$; and finally, if $t \prec -1$, put $w = a + (b - a)t$. No other cases need be considered since t is transcendental with respect to K and hence neither $E(0, t)$ nor $E(1, t)$ can be valid for S'. In each of these cases it is possible to show from the properties of an ordered field, which S' possesses by virtue of being a model for $\mathfrak{K}$, that $a \prec t \prec b$. Further, since in each case w is transcendental over S, $p(w, x)$ is irreducible over S and therefore irreducible over S' (see proof of Theorem 3.1), showing that w satisfies $I(z)$. This proves the theorem.

As final application, we will prove a theorem for fields K with a valuation ψ in an ordered field W. The valuations considered will be non-trivial and hence the function ψ will be such that **(7**, p. 325**)**:

3.21 for any k, $\psi(k)$ is an element of W,
3.22 for any $k \neq 0$, $0 < \psi(k)$, and $\psi(0) = 0$,
3.23 for any k and k', $\psi(k.k') = \psi(k).\psi(k')$,
3.24 for any k and k', $\psi(k+k') \leqslant \psi(k) + \psi(k')$,
3.25 there is a k such that $\psi(k) \neq 1$ and $\psi(k) \neq 0$.

The following properties can then be proven:

3.26 for any a, k and k', if $\psi(a) \neq 0$ and $\psi(k) < \psi(k')$, then $\psi(k \cdot a) < \psi(k' \cdot a)$ is true. This property follows from the fact that W is an ordered field, and from 3.23.
3.27 for any k, if $k \neq 0$ then $\psi(k) \neq 0$.
3.28 for any a, k and k', if $\psi(a) = \psi(1)$ and $\psi(k) < \psi(k')$, then $\psi(k \cdot a) < \psi(k')$.

THEOREM 3.3. *If K is a field with a valuation ψ in an ordered field W and K fulfils condition C, then for any a and b from K with $b \neq 0$ and for any irreducible polynomial $p(t, x)$ in x with coefficients in $K(t)$, there exists a t^* in K such that $\psi(a - t^*) < \psi(b)$ and such that $p(t^*, x)$ is irreducible in x over K.*

Proof. Let $E^*(x, y)$ and $Q^*(x, y)$ be the atomic predicates of $\mathfrak{L}$ which have been assigned to the following sets respectively: the set of pairs (a, b) of elements from K such that $\psi(a) = \psi(b)$, and the set of pairs (a, b) such that $\psi(a) < \psi(b)$. Then these predicates will appear in statements in $\mathfrak{K}$. For example, in addition to statements expressing that $E^*(x, y)$ is an equivalence relation, the following statements will appear in $\mathfrak{K}$:

$(x)(y)(z)(Q^*(x, y) \wedge Q^*(x, z) \supset Q^*(x, z))$,
$(x)(y)(Q^*(x, y) \vee Q^*(y, x) \supset E^*(x, y))$,
$(x)(y)(Q^*(x, y) \vee Q^*(y, x) \vee E^*(x, y))$,

together with statements corresponding to properties of the valuation function ψ, such as

3.31 $(x)(\sim E(x, 0) \supset Q^*(0, x)) \wedge (x)(E(x, 0) \supset E^*(x, 0))$,
3.32 $(\exists x)(\sim E^*(x, 0) \wedge \sim E^*(x, 1))$,
3.33 $(x)(y)(z)(u)(v)(Q^*(x,y) \wedge P(x,z,u) \wedge P(y,z,v) \wedge \sim E^*(z,0) \supset Q^*(u,v))$
3.34 $(x)(\sim E(x, 0) \supset \sim E^*(x, 0))$
3.35 $(x)(y)(z)(u)(Q(y, z) \wedge E^*(1, x) \wedge P(x, y, u) \supset Q^*(u, z))$,

where "0" and "1" are the individual parameters of $\mathfrak{L}$ corresponding to the zero and the unit of K.

If S' is the extension of K given by Theorem 2.1, then S' is a model for $\mathfrak{K}$. Hence in S' there must be defined relations corresponding to the atomic predicates E^*, Q^*, S and P. This can be expressed in the following way. The equivalence relation $E^*(x, y)$ will determine equivalence classes in S' for which $\bar{\psi}(a)$ for any member a of S' will denote the equivalence class determined by a. Thus for any a and b of S', $\bar{\psi}(a) = \bar{\psi}(b)$ if and only if $E^*(a, b)$ holds in S'. Further, these equivalence classes can be ordered by the relation corresponding to $Q^*(x, y)$, for we can say that for any a and b of S', $\bar{\psi}(a) < \bar{\psi}(b)$ if and only if $Q^*(a, b)$ holds in S', and this will be a proper ordering of the equivalence classes, by the statements listed above as appearing in K. Lastly, in S' will appear relations corresponding to S and P, and these relations will be the usual addition and multiplication relations of S' as in the proofs of Theorems 3.1 and 3.2. Since K is a subfield of S' and since the language $\mathfrak{L}$ contains parameters for all members of K, we can say that for any members a and b of K, $\bar{\psi}(a) = \bar{\psi}(b)$ if and only if $\psi(a) = \psi(b)$; i.e., that the equivalence classes determined in K by $\bar{\psi}$ are the same as those determined in K by ψ.

In order to prove the theorem, it is sufficient to show that for any parameters α and β of $\mathfrak{L}$ the statement

3.36 $(\exists z)(\sim E(\beta, 0) \supset (x)(y)(S(z, x, 0) \wedge S(\alpha, x, y) \supset Q^*(y, \beta)) \wedge I(z))$

holds in K, and hence that it holds in S'. It will hold in S' if and only if for every a and b in K, with $b \neq 0$, there is a w in S' such that $\bar{\psi}(a - w) < \bar{\psi}(b)$.

Now

(i) if $\bar{\psi}(t) < \bar{\psi}(1)$, let $w = a - b.t$;

(ii) if $\bar{\psi}(1) < \bar{\psi}(t)$, let $w = a - b.t^{-1}$; and

(iii) if $\bar{\psi}(t) = \bar{\psi}(1)$, choose k in K, $k \neq 0$, such that $\bar{\psi}(k) < \bar{\psi}(1)$, i.e., such that $\psi(k) < \psi(1)$, and let $w = a - b.k.t$.

In the last case, such a k can be found since 3.32 holds in K, and hence for some k, $k \neq 0$, either $\psi(k) < \psi(1)$ or $\psi(1) < \psi(k)$ and hence either $\psi(k) < \psi(1)$ or $\psi(k^{-1}) < \psi(1)$ by 3.26. Each of the w's chosen can be shown to satisfy $\bar{\psi}(a - w) < \bar{\psi}(b)$, for

(i) since $\sim E(b,0)$ holds in S' so does $\sim E^*(b,0)$ by 3.34, and therefore $\bar{\psi}(t.b) < \bar{\psi}(a)$ by 3.33;

(ii) By 3.26, $\bar{\psi}(t^{-1}) < \bar{\psi}(1)$; the remainder of the proof can be completed as in case (i);

(iii) $\bar{\psi}(k.b) < \bar{\psi}(b)$ can be proven following the previous cases, and from this by 3.35 can be proven $\bar{\psi}(k.b.t) < \bar{\psi}(b)$.

Finally, each of the w's will satisfy $I(z)$ since in each case $p(w, x)$ will be irreducible over S and therefore irreducible over S'. Hence we have shown that 3.36 holds in S' for any parameters α and β of $\mathfrak{L}$ and have therefore established the theorem.

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INDUCED AND PRODUCED MODULES

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Introduction. We shall consider here two generalizations to rings of the concept of induced representation as it occurs in the representation theory of finite groups **(6)**.

If A is a ring, S a subring of A, we shall associate with each S-module M an *induced pair* $(I(M), \kappa)$ consisting of an A-modulo $I(M)$ and an S-homomorphism $\kappa: M \to I(M)$. κ will be an isomorphism if and only if there exists an A-module having M as an S-submodule, and then $I(M)$ can be described as "the most general A-module generated by M"; so our definition is a generalization of the classical concept, consistent with that of other writers **(7; 11; 15)**.

Dually, we shall associate with M a *produced pair* $(P(M), \pi)$ consisting of an A-module $P(M)$ and an S-homomorphism $\pi: P(M) \to M$, the duality involved being an instance of the duality formalized by MacLane **(12)**.

We investigate certain dual properties of the induced and produced pairs. If M is given as an A-module, we can consider it as an S-module and form $(I(M), \kappa)$ and $(P(M), \pi)$. Then M is (A-isomorphic with) an A-quotient module of $I(M)$ and an A-submodule of $P(M)$. We determine when M is a direct summand of $I(M)$, or dually, of $P(M)$.

If A is a group ring, and S a subring by a subgroup of finite index, $I(M)$ and $P(M)$ are isomorphic A-modules for every unitary S-module M. The general question of when $I(M)$ and $P(M)$ are isomorphic leads to the consideration of certain generalized "Casimir operators" and to the notion of a ring *self-dual* with respect to a subring.

As applications of our general discussion we mention a characterization of separable algebras related to a result of Hochschild **(9)**, a characterization of the principal indecomposable representations of a Frobenius algebra, and certain theorems of Eckmann **(2)**, Gaschütz **(5)** and Higman **(7)**.

1. Induced and Produced Modules

1. The induced pair. Let S be a subring of a ring A. It is our intention to study some relations between representation modules for S and those for A. Our methods will not be complicated by extending the problem according to the following definitions, which also will be natural for the considerations of Part II:

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If S is a ring, a module M which is at the same time a left and a right S-module such that

$$s\cdot ut = su\cdot t \qquad (s, t \in S, u \in M)$$

is called an *S-bimodule*. We shall call a ring A an *S-ring* if it is an S-bimodule subject to the additional conditions

$$s\cdot ab = sa\cdot b,\quad a\cdot sb = as\cdot b,\quad a\cdot bs = ab\cdot s \qquad (a \in S; a, b \in A),$$

typical examples being the case in which S is a subring of A, or A a hypercomplex system over a commutative field S.

An S-bimodule M will be called a *right A,S-module* if it is a right A-module satisfying

$$s\cdot ua = su\cdot a,\quad u\cdot sa = us\cdot a,\quad u\cdot as = ua\cdot s \qquad (a \in S, u \in M, a \in A).$$

Interchanging left and right we obtain the definition of *left A,S-module*. We shall usually abbreviate "right A,S-module" to "A,S-module". Also, we shall usually state only one of a pair of left-right dual definitions or propositions.

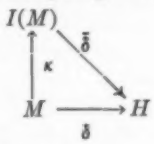
The S-ring A itself can be considered in a natural way as an A,S-module and as a left A,S module.

If S is a subring of A, a right A-module M becomes an A,S-module if we let S operate trivially on the left of M; $su = 0$ for all $s \in S$, $u \in M$.

We write $M \simeq {}_AN$ to indicate the existence of an A,S-isomorphism of an A,S-module M onto an A,S-module N.

We shall now consider certain A,S-modules determined by S-bimodules. We claim that to each S-bimodule M there corresponds a pair $(I(M), \kappa)$, consisting of an A,S-module $I(M) = I_{A,S}(M)$ and an S-homomorphism $\kappa: M \to I(M)$, determined up to an A,S-isomorphism by the property:

(I) *If H is any A,S-module and $\delta: M \to H$ is an S-homomorphism then there exists one and only one A,S-homomorphism $\bar{\delta}: I(M) \to H$ such that $\delta = \kappa\bar{\delta}$, in other words, such that the following diagram is commutative:*



We shall say that the pair $(I(M), \kappa)$ is *induced* by M, writing $I(M) = I_{A,S}(M)$ when we wish to stress the role of A and S.

We postpone to §7 the exhibition of a pair satisfying (I). That there can exist essentially only one such pair is readily seen as follows: Suppose that the pair (H, δ) also satisfies (I). Then there exists by (I) an A,S-homomorphism $\bar{\delta}: I(M) \to H$ such that $\delta = \kappa\bar{\delta}$, and an A,S-homomorphism $\bar{\kappa}: H \to I(M)$ such that $\kappa = \delta\bar{\kappa}$. Then $\bar{\delta}\bar{\kappa}$ is an A,S-endomorphism of $I(M)$ and $\kappa\bar{\delta}\bar{\kappa} = \delta\bar{\kappa} = \kappa$. It follows from the uniqueness requirement of (I) that $\bar{\delta}\bar{\kappa}$ is the identity automorphism of $I(M)$. By symmetry $\bar{\kappa}\bar{\delta}$ is the identity automorphism of H, proving that $\bar{\delta} = \bar{\kappa}^{-1}$ is an A,S-isomorphism of $I(M)$ onto H, with $\delta = \kappa\bar{\delta}$.

Another characterization of the induced pair is as follows: We say that a pair (H, δ), consisting of an A,S-module H and an S-homomorphism $\delta: M \to H$, is *generated* by M if the smallest A,S-submodule of H containing $M\delta$ is H itself. If δ is an isomorphism, we say that H is *generated* by M. We have

THEOREM 1. *The induced pair $(I(M), \kappa)$ is determined up to an A,S-isomorphism by the properties*

(a) *$(I(M), \kappa)$ is generated by M.*

(b) *if (H, δ) is generated by M then there exists an A,S-homomorphism $\bar{\delta}: I(M) \to H$ such that $\delta = \kappa\bar{\delta}$.*

For the proof we use

LEMMA 1. *Let (H, δ) be generated by M and let κ be an A,S-module, $\epsilon: M \to K$ an S-homomorphism. Then there exists at most one A,S-homomorphism $\bar{\imath}: H \to K$ such that $\epsilon = \delta\bar{\imath}$.*

Proof. The equation $\epsilon = \delta\bar{\imath}$ means that the restriction of $\bar{\imath}$ to $M\delta$ is uniquely determined by ϵ. Since $\bar{\imath}$ is an A,S-homomorphism, we have for all a in A that $\delta a \cdot \bar{\imath} = \delta\bar{\imath} \cdot a = \epsilon a$, which means that the restriction of $\bar{\imath}$ to $M \cdot \delta a$ is uniquely determined by ϵ. Hence $\bar{\imath}$ is uniquely determined by ϵ since H is the l.u.b. of its submodules $M\delta$ and $M \cdot \delta a$ with a in A.

Proof of Theorem 1. That $(I(M), \kappa)$ satisfies (b) follows at once from its defining property (I).

To prove (a) we show that the g.l.b. $\bar{M}$ of the A,S-submodules of $I(M)$ which contain $M\kappa$ coincides with $I(M)$. Let j be the injection of M into $I(M)$. Then $\kappa = \alpha j$ defines an S-homomorphism $\alpha: M \to \bar{M}$. Since (M, α) is generated by M, as is clear from its construction, Lemma 1 implies that j is the only A,S-homomorphism of M into $I(M)$ such that $\kappa = \alpha j$. By (I) there exists an A,S-homomorphism $\bar{\alpha}: I(M) \to \bar{M}$ such that $\alpha = \kappa\bar{\alpha}$. Then $\kappa = \alpha j = \kappa\bar{\alpha}j = \alpha j\bar{\alpha}j$. But $j\bar{\alpha}j: \bar{M} \to I(M)$ is an A,S-homomorphism. Hence $j\bar{\alpha}j = j$, so that $j\bar{\alpha}$ is the identity automorphism of $I(M)$. Similarly $\bar{\alpha}j$ is the identity automorphism of $\bar{M}$, proving that $\bar{\alpha} = j^{-1}$, and hence that $\bar{M} = I(M)$. Consequently (a) is satisfied by the induced pair.

That $(I(M), \kappa)$ is essentially uniquely determined by (a) and (b) is readily verified using Lemma 1, thus completing the proof of Theorem 1.

When κ is an isomorphism, so that M can be identified with the S-submodule $M\kappa$ of $I(M)$, we say the $I(M)$ *is induced by* M. By (I) we see that this is the case if and only if there exists an A,S-module having M as an S-submodule. By Theorem 1, when κ is an isomorphism we can describe $I(M)$ as "the most general A,S-module generated by M", and this is the only case in which there exists an A,S-module conforming to this description. Thus our definition of induced module is consistent with the classical one; for instance, if A is a ring with identity element e, S a subring of A containing e, and R a right ideal of S (turned into an S-bimodule by letting S operate trivially on the left) then $I(R) = RA$.

2. The duality and the produced pair. We want to subject our discussion to a dualization, the duality involved being an instance of the duality formalized by MacLane **(12)**.

If M and N are modules, we write

$$\alpha: M \to N$$

to mean that α is a homomorphism of M into N, and write $K(\alpha)$ for the kernel of α. If M is a submodule of N, $u \to u$ $(u \in M)$ defines the *injection* of M into N, while if M is a quotient module of N, $M = N/L$, $v \to L + v$ $(v \in N)$ defines the *projection* of N onto M.

To any statement $\mathfrak{S}$ about modules, referring to homomorphisms of one such into another, submodules, quotient modules, injections and projections, but not to elements of modules, we assign a statement $\overline{\mathfrak{S}}$ by carrying out in $\mathfrak{S}$ the interchanges indicated by the following table:

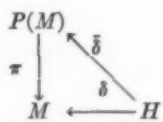
$\alpha: M \to N$	$\alpha: N \to M$
α an isomorphism into product $\alpha\beta$	α a homomorphism onto product $\beta\alpha$
submodule	quotient module
injection	projection

Note that "S-bimodule" remains unchanged, while "right A,S-module" is interchanged with "left A,S-module".

In the terminology of MacLane **(12)**, $\overline{\mathfrak{S}}$ is the dual of $\mathfrak{S}$. We shall call the left-right dual $\mathfrak{S}'$ of $\overline{\mathfrak{S}}$ the *dual* of $\mathfrak{S}$. Then "right A,S-module" and "left A,S-module" are self dual.

In particular, the dual of the (right) A,S-module $I(M)$ is an A,S-module. Precisely, the dual of our statement defining and asserting the existence and essential uniqueness of the induced pair is the following (as before, S is a ring, A an S-ring):

To each S-bimodule M there corresponds a pair $(P(M), \pi)$, consisting of an A,S-module $P(M)$ and an S-homomorphism $\pi: P(M) \to M$, determined up to an A,S-isomorphism by the property (P) *if H is any A,S-module, and $\delta: H \to M$ is an S-homomorphism, then there exists one and only one A,S-homomorphism $\bar{\delta}: H \to P(M)$ such that $\delta = \bar{\delta}\pi$*:



We call $(P(M), \pi)$ the pair *produced* by M. Its existence will be established in §7 by the exhibition of a pair satisfying (P). The essential uniqueness of the produced pair can be shown by dualizing the demonstration of the corresponding fact for the induced pair, i.e., by replacing each statement of that demonstration by its dual in the sense defined above.

Actually, the uniqueness proof for $(I(M), \kappa)$ can easily be translated into a "categorical" one for the abelian bicategory of all modules and homomorphisms of modules into modules **(12)**. Hence we can infer at once the essential uniqueness of $(P(M), \pi)$ from that of $(I(M), \kappa)$ by invoking the *duality principle* of MacLane **(12**, p. 806**)**.

Except for the existence proofs for $(I(M), \kappa)$ and $(P(M), \pi)$, the same remark will apply to all the theorems of this Part I. These will occur in dual pairs $\mathfrak{S}$, $\mathfrak{S}'$, of which we shall prove only one, say $\mathfrak{S}$, phrasing the proof in such a way that it can easily be translated into a categorical one. Then we can conclude at once from MacLane's duality principle that $\overline{\mathfrak{S}}$, and hence $\mathfrak{S}'$ is a theorem. If we do not wish to refer to the duality principle, we can obtain a proof of $\mathfrak{S}'$ by dualizing that of $\mathfrak{S}$.

Dualizing the definition of generated pair as given in §1, we say that a pair (H, δ), consisting of an A,S-module H and an S-homomorphism $\delta: H \to M$, is *permeated* by the S-bimodule M if the only A,S-quotient module of H containing $H/K(\delta)$ is H itself. The reader will readily obtain a characterization of the produced pair in terms of this notion by dualizing Theorem 1.

When π is onto, so that M can be identified with the S-quotient module $P(M)/K(\pi)$ of $P(M)$, we say that $P(M)$ is *produced* by M, thus dualizing the notion of induced module. According to (P), π is onto if and only if there exists an A,S-module of which M is an S-quotient module.

3. Iteration and direct sums. When M is given as a right S-module, it can be turned into an S-bimodule by letting S operate trivially on the left; $su = 0$ for all s in S, u in M. Then it is easily seen that S operates trivially on the left of the A,S-modules $I(M)$ and $P(M)$.

Suppose that T is a subring of S, then the S-ring A is also a T-ring. Let M be a right T-module and consider the induced pairs $(I_{S,T}(M), \kappa_T)$ and $(I_{A,S}[I_{S,T}(M)], \kappa_S)$. Then it is seen by verifying (I) that $(I_{A,S}[I_{S,T}(M)], \kappa_T\kappa_S)$ is induced by M. We have the formula

$$I_{A,S}\, I_{S,T} = I_{A,T} \tag{3}$$

and dually

$$P_{A,S}\, P_{S,T} = P_{A,T}. \tag{3'}$$

Similarly it is proved that if an S-bimodule M is a direct sum $M = M_1 \oplus M_2$ of S-submodules, then we may choose

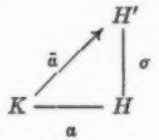
$$I(M_1 \oplus M_2) = I(M_1) \oplus I(M_2) \tag{4}$$

$$P(M_1 \oplus M_2) = P(M_1) \oplus P(M_2). \tag{4'}$$

4. A,S-injective and A,S-projective modules. Let H be an A,S-module. We shall call H an *A,S-protract* of an A,S-module H' if there exists commutative diagram

where $\sigma: H' \to H$ is an A,S-homomorphism, $\epsilon: H \to H'$ is an S-homomorphism, and i is the identity automorphism of H. This implies that σ is onto, $H \simeq_A H'/K(\sigma)$, ϵ is an isomorphism, and, as an S-bimodule, $H' = K(\sigma) \oplus H\epsilon$. We shall also refer to H as an A,S-protract of (H', σ) or of (H', σ, ϵ).

An A,S-module K will be called *A,S-projective* if, whenever α is an A,S-homomorphism of K into an A,S-module H, and H is an A,S-protract of (H', σ), then there exists an A,S-homomorphism $\bar{\alpha}: K \to H'$ such that $\alpha = \bar{\alpha}\sigma$:

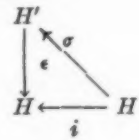


The interest of this definition for our present purposes arises from the following fact:

THEOREM 2. *For any S-bimodule M, the A,S-module $I(M)$ is A,S-projective.*

Proof. Let H be an A,S-protract of (H', σ, ϵ) and let $\alpha: I(M) \to H$ be an A,S-homomorphism. Now $\phi = \kappa\alpha\epsilon: M \to H'$ is an S-homomorphism, hence by (I) there exists an A,S-homomorphism $\bar{\alpha}: I(M) \to H'$ such that $\phi = \kappa\bar{\alpha}$. Then $\kappa\bar{\alpha}\sigma = \phi\sigma = \kappa\alpha\epsilon\sigma = \kappa\alpha$. But $\bar{\alpha}\sigma: I(M) \to H'$ is an A,S-homomorphism, hence by the uniqueness part of (I), $\alpha = \bar{\alpha}\sigma$, proving that $I(M)$ is A,S-projective.

From the notion of A,S-protract we obtain by dualization that of *A,S-retract*; that an A,S-module H is an A,S-retract of an A,S-module H' means the existence of a commutative diagram



where $\sigma: H \to H'$ is an A,S-homomorphism, $\epsilon: H' \to H$ is an S-homomorphism, and as before, i is the identity automorphism of H. Then σ is an isomorphism, ϵ is onto, and, as an S-bimodule, $H' = H\sigma \oplus K(\epsilon)$. We shall also refer to H as an A,S-retract of (H', σ), or of (H', σ, ϵ).

Dually to A,S-projective modules we have *A,S-injective* modules, an A,S-module K being called A,S-injective if, whenever H is an A,S-retract of

(H', σ), and $\alpha: H \to K$ is an A,S-homomorphism, there exists an A,S-homomorphism $\bar{\alpha}: H' \to K$ such that $\alpha = \sigma\bar{\alpha}$. Dual to the fact that $I(M)$ is A,S-projective we have

THEOREM 2'. *For any S-bimodule M, $P(M)$ is A,S-injective.*

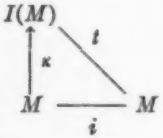
5. The case M an A,S-module. Now we consider the case in which M is given as an A,S-module to begin with. Then we can consider M as an S-bimodule in order to form the induced pair $(I(M), \kappa)$ and the produced pair $(P(M), \pi)$.

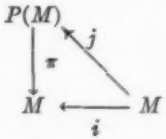
Let i be the identity automorphism of M. By (I) there exists one and only one A,S-homomorphism $t: I(M) \to M$ such that $i = \kappa t$. It follows that κt must be an isomorphism and t must be onto. We have

THEOREM 3. *If M is an A,S-module, then $I_{A,S}(M)$ is induced by M (considered as an S-bimodule), and there exists one and only one A,S-homomorphism of $I(M)$ onto M such that M is an A,S-protract of $(I(M), t, \kappa)$.*

Dually

THEOREM 3'. *$P_{A,S}(M)$ is produced by M, and there exists one and only one A,S-isomorphism $j: M \to P(M)$ such that M is an A,S-retract of $(P(M), j, \pi)$.*





Thus an A,S-module M is always an A,S-protract of $I(M)$ and an A,S-retract of $P(M)$. When is it isomorphic with a direct summand of one of these A,S-modules? We shall prove[1]

THEOREM 4. *For an A,S-module M, the following conditions imply each other:*

(a) *$K(t)$ is a direct summand of $I_{A,S}(M)$, so that $I_{A,S}(M) \simeq {}_AK(t) \oplus M$.*
(b) *M is (A,S-isomorphic with) a direct summand of $I_{A,S}(M)$.*
(c) *M is A,S-projective.*
(d) *if M is an A,S-protract of (H, σ), then $K(\sigma)$ is a direct summand of H, so that $H \simeq K(\sigma) \oplus M$.*
(e) *M is A,S-isomorphic with a direct summand of every A,S-protract of M.*

Proof. (a) implies (b) at once.

(b) implies (c): By Theorem 2, $I(M)$, and hence every direct summand thereof, is A,S-projective. Hence (b) implies (c).

[1]The reader will readily see the relation of our result to results of Baer (1) and Eckmann and Schopf (4).

(c) implies (d): Suppose M is an A,S-retract of (H, σ). Let i be the identity automorphism of M. Then (c) implies the existence of an A,S-homomorphism $\bar{\imath}: M \to H$ such that $\bar{\imath}\sigma = i$. Hence $\bar{\imath}$ is an isomorphism and $H = K(\sigma) \oplus M\bar{\imath}$, proving that (c) implies (d).

(d) implies (e) at once.

(e) implies (a): By Theorem 3, M is an A,S-protract of $I(M)$. Hence (e) implies that M is a direct summand of $I(M)$, so that M is A,S-projective by Theorem 2. By Theorem 3, M is an A,S-protract of $(I(M), t)$, hence $K(t)$ is a direct summand of $I(M)$ by the implication of (c) by (d) already proved. Thus (e) implies (a), completing the proof of Theorem 4.

The dual of Theorem 4 is

THEOREM 4′. *For an A,S-module M the following are mutually equivalent conditions:*

(a′) *Mj is a direct summand of $P_{A,S}(M)$, so that $P_{A,S}(M) \simeq {}_AP(M)/Mj \simeq M$.*

(b′) *M is (A,S-isomorphic with) a direct summand of $P_{A,S}(M)$.*

(c′) *M is A,S-injective.*

(d′) *if M is an A,S-retract of (H, σ), then $M\sigma$ is a direct summand of H, so that $H \simeq {}_AH/M\sigma \oplus M$.*

(e′) *M is (A,S-isomorphic with) a direct summand of every A,S-retract of M.*

COROLLORY. *Let H be an A,S-projective (injective), indecomposable A,S-module such that the S-submodules of H and the A,S-submodules of $I_{A,S}(H)$ ($P_{A,S}(H)$) satisfy the descending chain condition. Then there exists an indecomposable S-bimodule M such that $I_{A,S}(M)$ is induced by M($P_{A,S}(M)$ is produced by M) and contains a direct summand A,S-isomorphic with H.*

Proof. As an Bimodule, H has a Remak decomposition $H = M_1 \oplus \ldots \oplus M_k$, M_i an indecomposable S-bimodule. By (4), $I(H) \simeq {}_AI(M_1) \oplus \ldots \oplus I(M_k)$, and this decomposition can be refined to a Remak decomposition. If H is A,S-projective, we infer from Theorem 4 that H is A,S-isomorphic with a direct summand of $I(H)$. Hence, since H is indecomposable, it is A,S-isomorphic with a direct summand of $I(M_i)$ for some i by the Remak theorem. $I(M_i)$ is induced by M since $I(H)$ is induced by H. Thus the corollary is proved with $M = M_i$.

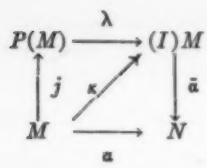
6. The Casimir operators. Now assume the existence of an A,S-homomorphism

$$\lambda: P_{A,S}(M) \to I_{A,S}(M)$$

for some fixed A,S-module M. Note that this assumption is self dual.

If N is a second A,S-module, and $\alpha: M \to N$ is an S-homomorphism, there exists by (I) one and only one A,S-homomorphism $\bar{\alpha}: I(M) \to N$ such that $\alpha = \kappa\bar{\alpha}$. If $j: M \to P(M)$ is the A,S-isomorphism of Theorem 3′, then

(5) $\hat{\alpha} = j\lambda\bar{\alpha}: M \to N$ is an A,S-homomorphism.



Dually, if $\beta: N \to M$ is an A,S-homomorphism,

(5′) $\tilde{\beta} = \tilde{\beta}\lambda t: N \to M$

is an A,S-homomorphism, where $\tilde{\beta}: N \to P(M)$ is the one and only A,S-homomorphism such that $\beta = \tilde{\beta}\pi$, whose existence is guaranteed by (P), and $t: I(M) \to P(M)$ is the homomorphism onto of Theorem 3.

The mappings $\hat{\alpha}$ and $\tilde{\beta}$ are generalizations of the Casimir operators which occur in the theory of separable algebras as will be seen in §8. If α and β are A,S-endomorphisms of an A,S-module M, then $\hat{\alpha}$ and $\tilde{\beta}$ are A,S-endomorphisms of M. We have the following self-dual theorem:

THEOREM 5. *Let i be the identity automorphism of M. Then each of the conditions*

(1) *There exists an S-endomorphism α of M such that $\hat{\alpha} = i$.*

(2) *There exists an S-endomorphism β of M such that $\tilde{\beta} = i$.*

Implies all the conditions (a) *through* (e) *and* (a′) *through* (e′) *of Theorems* 4, 4′.

If λ is an isomorphism onto, the conditions (a) *through* (e) *and* (a′) *through* (e′) *are mutually equivalent and imply* (1) *and* (2).

Proof. If $\hat{\alpha} = j\lambda\bar{\alpha} = i$ then $\lambda\bar{\alpha}: P(M) \to M$ is an A,S-homomorphism onto and $P(M) = Mj \oplus K(\lambda\bar{\alpha})$, proving (a′) of Theorem 4′. Furthermore, $j\lambda: M \to I(M)$ is an A,S-isomorphism, $\bar{\alpha}: I(M) \to M$ is an A,S-homomorphism onto, and $I(M) = Mj\lambda \oplus K(\bar{\alpha})$, proving (b) of Theorem 4. The remaining conditions of Theorems 4 and 4′ follow from those theorems.

Next assume that λ is an isomorphism onto. That the conditions (a) through (e′) are mutually equivalent follows at once from this assumption and Theorems 4, 4′. If we now assume (a′) of Theorem 4′, i.e., that Mj is a direct summand of $P(M)$, then there exists an A,S-homomorphism $\bar{\imath}: P(M) \to M$ such that $j\bar{\imath} = i$. Then $\alpha = \kappa\lambda^{-1}\bar{\imath}$ is an S-endomorphism of M, and $\bar{\alpha} = \lambda^{-1}i$. Hence $\hat{\alpha} = j\lambda\bar{\alpha} = j\bar{\imath} = i$, proving (1).

The rest of Theorem 5 is dual to the part already proved.

7. Construction of induced and produced pairs. For the construction of the induced and produced pairs we need the following definitions, also used in later sections.

Let M be an S-bimodule. By $M \otimes_S A$ we denote the tensor product of M and A over S, that is, the module generated by all pairs $u \otimes a$, with u in M, a in A, with the relations

$$us \otimes a = u \otimes sa \qquad (s \in S)$$

$$(u_1 + u_2) \otimes a = u_1 \otimes a + u_2 \otimes a, \quad u \otimes (a_1 + a_2) = u \otimes a_1 + u \otimes a_2$$

into an A,S-module by

$$s(u \otimes a) = su \otimes a, \quad (u \otimes a)s = u \otimes as,$$

$$(u \otimes a)b = u \otimes ab \qquad (s \in S, b \in A).$$

By $S\text{-Hom}_r(A, M)$ we shall denote the module of all module homomorphisms $f: A \to M$ such that

$$as \cdot f = af \cdot s \qquad (s \in S)$$

turned into an A,S-module by

$$a(s^*f) = s(af), \quad a(f^*s) = (sa)f,$$

$$a(f^*b) = (ba)f \qquad (s \in S, b \in A).$$

For the construction of the induced pair let us assume first that A is a ring with identity element e, S a subring of A containing e, and let the S-bimodule M be *right unitary*, i.e., such that $ue = u$ for all u in M. In this case we can choose

$$I_{A,S}(M) = M \otimes {}_SA, \quad \kappa: u \to u \otimes e \qquad (u \in M)$$

for the induced pair. For we can check that if H is any A,S-module and $\delta: M \to H$ is an S-homomorphism, then

$$\bar{\delta}: u \otimes a \to u\delta \cdot a \tag{6}$$

determines the one and only one A,S-homomorphism $\bar{\delta}: I(M) \to H$ such that $\delta = \kappa\bar{\delta}$.

Furthermore, in this case we can choose

$$P_{A,S}(M) = S\text{-Hom}_r(A, M), \quad \pi: f \to ef \quad (f \in S\text{-Hom}_r(A, M)$$

as a realization of the produced pair. If $\delta: H \to M$ is an A,S-homomorphism,

$$a \cdot h\delta = ha \cdot \delta \qquad (h \in H) \tag{6'}$$

defines the one and only A,S-homomorphism $\bar{\delta}: H \to S\text{-Hom}_r(A, M)$ such that $\delta = \bar{\delta}\kappa$.

In particular, if M is given as an A,S-module, then the A,S-homomorphism onto $t: I(M) \to M$ of Theorem 3 is given by

$$t: u \otimes a \to ua \tag{7}$$

while the A,S-isomorphism into $j: M \to P(M)$ of Theorem 3′ is given by

$$a \cdot uj = ua. \tag{7'}$$

The general case of an S-ring A and an S-bimodule M is easily reduced to the one just considered.

II. Self-Dual S-Rings

8. S-bilinear forms. For an S-ring A, $A' = S\text{-Hom}_r(A, S)$ as defined in §7 is an A,S-module. Let $\phi: A \to A'$ be an A,S-homomorphism, (considering A as an A,S-module). Then

$$(a, b) = b \cdot a\phi \qquad (a, b \in A) \tag{1}$$

defines a mapping (a, b) of the set of pairs of elements of A into S with the properties

$$\begin{aligned} &(a_1 + a_2, b) = (a_1, b) + (a_2, b),\ (a, b_1 + b_2) = (a, b_1) + (a, b_2) \\ &(as, b) = (a, sb),\ s(a, b) = (sa, b),\ (a, b)s = (a, bs) \qquad (s \in S) \end{aligned} \tag{2}$$

and

$$(ab, c) = (a, bc) \qquad (b \in A). \tag{3}$$

Such a mapping satisfying conditions (2) is called an *S-bilinear form* on A. If in addition it satisfies (3) we call it *associative*. A 1.1 correspondence between the S-bilinear forms (a, b) on A and the A,S-homomorphisms $\phi: A \to A'$ is established by (1).

Interchanging left and right in the definition of A', we define $A'' = S\text{-Hom}_l(A, S)$ as the module of all module homomorphisms $f: M \to S$ such that $sa \cdot f = s \cdot af$ $(s \in S, a \in A)$, turned into a left A,S-module by

$$a(f^*s) = (af)s,\ a(s^*f) = (sa)f,\ a(b^*f) = (ba)f \qquad (s \in S, a, b \in A).$$

By the symmetry of the conditions (2) and (3) we have that

$$(a, b) = a \cdot b\phi' \tag{4}$$

defines an A,S-homomorphism $\phi': A \to A''$.

Note that ϕ is an isomorphism if and only if

$$(a, A) = 0 \text{ implies } a = 0 \tag{5}$$

while ϕ' is an isomorphism if and only if

$$(A, a) = 0 \text{ implies } a = 0. \tag{6}$$

An S-bilinear form satisfying (5) and (6) is called *non-singular*.

We shall call an S-bilinear form (a, b) on A *invariant* if the A,S-homomorphisms $\phi: A \to A'$ and $\phi': A \to A''$ defined by (1) and (4) respectively are both isomorphisms onto.

An S-ring A will be called *self-dual* (with respect to S) if it admits an invariant S-bilinear form. Note that if A is an algebra over a field S, this condition implies that the dimension of A over S is finite and hence is much stronger than the assumption of the existence of a non-singular form.

Let $[a, b]$ be a second invariant S-bilinear form on A determining A,S-isomorphisms $\psi: A \to A'$, $\psi': A \to A''$ according to (1) and (4). Then

$$[a, b] = (a\sigma, b) = (a, b\sigma') \tag{7}$$

where $\sigma = \psi\phi^{-1}$ and $\sigma' = \psi'\phi'^{-1}$ are A,S-automorphisms of A considered as a right and a left A,S-module respectively.

9. Casimir operators. By a *left S-basis* of an S-ring A we shall mean a finite set $a_1, \ldots, a_n$ of elements of A such that each a in A can be written uniquely in the form

$$a = \sum_{j=1}^{n} s_j a_j \qquad (s \in S)$$

i.e., $a_1, \ldots, a_n$ is a finite free basis of A as a left S-module. *A right basis* is defined similarly.

Throughout §9 we shall assume that A is a ring with identity element e, S a subring of A containing e. If then M is an S-bimodule such that $ue = u$ for all u in M (called right unitary), we can, according to §7, take $I_{A,S}(M) = M \otimes_S A$ and $P_{A,S}(M) = S\text{-Hom}_r(A, M)$. In particular, $A = I(S)$ and $A' = P(S)$.

If A has both a left and a right S-basis, and if there exists an associative S-bilinear form (a, b) on A such that the A,S-homomorphism $\phi: A \to A'$ defined by (1) is an A,S-isomorphism onto, we see that to each left S-basis $a_1, \ldots, a_n$ of A there corresponds a right S-basis $\bar{a}_1, \ldots, \bar{a}_n$ of A determined by

$$(a_i, \bar{a}_j) = \delta_{ij} \tag{8}$$

where $\delta_{ij} = 0$ if $i \neq j$, $\delta_{ii} = e$. Consequently (a, b) is invariant and A is self-dual with respect to S.

Given an invariant S-bilinear form (a, b) on A, a left S-basis $a_1, \ldots, a_n$ and a right S-basis $\bar{a}_1, \ldots, \bar{a}_n$ satisfying (8) will be called *dual*. By (8) and the associativity of (a, b) we have

$$a_i a = \sum_{j=1}^{n} a_j s_{ji}, \ (a \in A, s_{ji} \in S) \text{ implies } a\bar{a}_i = \sum_{j=1}^{n} a_i \bar{a}_j. \tag{9}$$

One proves easily

LEMMA 2. *If $a_1, \ldots, a_n$ is a left S-basis, $\bar{a}_1, \ldots, \bar{a}_n$ a set of elements of A satisfying* (9), *then*

$$\lambda: f \to \sum_{i=1}^{n} \bar{a}_i f \otimes a_i \tag{10}$$

defines an A,S-homomorphism $\lambda: P(M) \to I(M)$ for any right unitary S-bimodule M. If $\bar{a}_1, \ldots, \bar{a}_n$ is a right S-basis of A, then λ is an isomorphism onto, and A is self-dual.

Combining this with the above remarks we obtain

PROPOSITION 1. *Let A be a ring with identity element e, S a subring of A containing e. If A has both a left and a right S-basis, each of the following conditions is necessary and sufficient for A to be self-dual:*

(a) $A \simeq {}_AA' = S\text{-Hom}_r(A, S)$.

(b) $I_{A,S}(M) \simeq {}_AP_{A,S}(M)$ *for every right unitary S-bimodule M.*

(c) *to each left S-basis $a_1, \ldots, a_n$ of A there corresponds a right S-basis $\bar{a}_1, \ldots, \bar{a}_n$ satisfying* (9).

Now we compute the Casimir operators of §6 which are determined by the mapping λ defined by (10). Accordingly, we assume that A is an S-ring having a left S-basis $a_1, \ldots, a_n$ and a set of elements $\bar{a}_1, \ldots, \bar{a}_n$ satisfying (9). Let M and N be right unitary A,S-modules, and let $\alpha: M \to N$ be an S-homomorphism, $\hat{\alpha}: M \to N$ the corresponding A,S-homomorphism determined by $\lambda: P(M) \to I(M)$ according to (5) of Part I.

By (5), (6′), and (7) of Part I, and (10) we have for u in M

$$u\hat{\alpha} = uj\lambda\bar{\alpha} = (\sum \bar{a}_i \cdot uj \otimes a_i)\bar{\alpha} = (\sum u\bar{a}_i \otimes a_i)\bar{\alpha} = \sum u\bar{a}_i \alpha a_i$$

i.e.,

$$\bar{\alpha} = \sum_{i=1}^{n} \hat{\alpha}_i \alpha a_i. \tag{11}$$

Similarly, if $\beta: N \to M$ is an S-homomorphism and $\tilde{\beta}: N \to M$ is the corresponding A,S-homomorphism given by (5′) of Part I, we have

$$\tilde{\beta} = \sum \bar{a}_i \beta a_i = \hat{\beta}. \tag{12}$$

These formulas (11) and (12) justify our reference to $\hat{\alpha}$ and $\tilde{\beta}$ as generalized Casimir operators. Let A be a separable algebra over a field S, and let $a_1, \ldots, a_n$ be a basis of A, $\bar{a}_1, \ldots, \bar{a}_n$ a dual basis of A with respect to a discriminant matrix. If α is a linear transformation of a representation module for A over S, $\hat{\alpha}$ is the Casimir operator of classical theory **(8)**.

By (11) and Theorem 5 we have

THEOREM 6. *Let A be a ring with identity element e, S a subring of A containing e. Suppose that there exists a left S-basis $a_1, \ldots, a_n$ of A and a set of elements $\bar{a}_1, \ldots, \bar{a}_n$ of A satisfying* (9). *Let M be a right unitary A,S-module. Then the condition*

(*) *There exists an S-homomorphism α of M such that*

$$\sum_{i=1}^{n} \bar{a}_i \alpha a_i = i$$

(the identity automorphism of M) is sufficient for all the conditions (a) *through* (e′) *of Theorems* 4, 4′. *If $\bar{a}_1, \ldots, \bar{a}_n$ is a right S-basis of A, each of these conditions is equivalent to* (*).

This theorem contains results of Eckmann, Gaschütz and the author (cf. §13).

10. The case of commutative S. *Throughout* §10 *we shall assume that S is a commutative ring, and that A is a self-dual S-ring such that $sa = as$ for all $a \in A$, $s \in S$.* In this case, considered as S-bimodules, $\mathrm{Hom}_r(A,S) = \mathrm{Hom}_l(A,S)$. The assumption that A is self-dual with respect to S means that there exists an invarient S-bilinear form (a, b) on A, determining A,S-isomorphisms ϕ of A onto $\mathrm{Hom}_r(A,S)$ and ϕ' of A onto $\mathrm{Hom}_l(A,S)$. Since ϕ and ϕ' both have the same range, there exists for each a in A elements $a\dagger$ and a^* in A such that $a^*\phi = a\phi'$ and $a\phi = a\dagger\phi'$, i.e., such that

$$(a^*, b) = (b, a) \text{ and } (a, b) = (b, a\dagger). \tag{13}$$

Since (a, b) is invariant,

$$\phi^*: a \to a^* \text{ and } \phi\dagger: a \to a\dagger \tag{14}$$

are reciprocal S-automorphisms of the S-bimodule A. Furthermore,

$$((ab)^*, c) = (c, ab) = (ca, b) = (b^*c, a) = (a^*, b^*c) = (a^*b^*, c),$$

which implies that $(ab)^* = a^* b^*$. This proves that ϕ^* is an automorphism off the S-ring A, the inverse of ϕ^* being $\phi\dagger$.

Let $[a, b]$ be a second invariant S-bilinear form on A, and ψ the corresponding A,S-isomorphism of A onto $\mathrm{Hom}_r(A,S)$. The relation between ϕ^* and ψ^* is readily seen from (7). Let us consider in particular the case in which A has an identity element, then the A,S-automorphisms of A considered as an A,S-module are the left multiplications by units of A. In particular $\sigma = \psi\phi^{-1}$ is effected by left multiplication by a unit, say c_0, of A, and hence by (7)

$$[a, b] = (c_0 a, b). \tag{15}$$

Hence $[a\phi\dagger\psi^*, b] = [b, a\phi\dagger] = (c_0 b, a\phi\dagger) = (a, c_0 b) = [c_0^{-1}a, c_0 b] = [c_0^{-1}ac_0, b]$, so that $\phi\dagger\psi^*$ is the inner automorphism of A induced by c_0. It is easy to see that all the inner automorphisms of A are obtained in this way from invariant S-bilinear forms, hence

PROPOSITION 2. *If A has an identity element, the automorphisms ϕ^* of A determined by invariant S-bilinear forms on A according to* (13) *and* (14) *constitute a full residue class modulo the inner automorphisms of A.*

COROLLARY. *There exists a symmetric invariant S-bilinear form on A if and only if there exists an invariant S-bilinear form on A such that the corresponding automorphism ϕ^* is inner, and then all the automorphisms of A determined by invariant S-bilinear forms are inner.*

Proposition 3 and its Corollary were obtained by Nakayama for Frobenius algebras **(14)**.

Suppose that A possesses a right identity element e. Then $(a, b) = (a, be) = (e^*, ab) = (e^*a, b)$, so that $a = e^*a$ and e^* is a left identity. Hence $e = e^*$ is an identity element for A.

LEMMA 2. *If A possesses a one sided identity element then it possesses an identity element.*

Now we shall record some further criteria for the existence of an identity element in A, assuming the descending chain condition for left S-ideals. We shall use the following notation: For a subset X of A,

$$p(X) = \{a \in A | (X, a) = 0\}$$
$$q(X) = \{a \in A | (a, X) = 0\}$$

Then p and q induce lattice endomorphisms of the lattice of submodules of A. If L is a left ideal, the associativity of (a, b) implies that $p(L)$ is a right ideal. Also by (13), $q(L) = p(L\dagger)$ so that $q(L)$ is also a right ideal. Hence p and q induce homomorphisms of the lattice of left S-ideals into the lattice of right S-ideals. By symmetry, they also induce lattice homomorphisms in the opposite direction.

From the definitions of p and q we have $p(X) \subseteq$ the right annihilator $I_r(X)$ of X, and $q(X) \subseteq$ the left annihilator $I_l(X)$. If A has an identity we have equality.

PROPOSITION 3. *If the left S-ideals of A satisfy the descending chain condition, each of the following conditions is necessary and sufficient for the existence of an identity element in A. N denotes the radical of the S-ring A.*

(a) *$I_r(L) = p(L)$ for every nilpotent minimal left S-ideal L, and $I_r(N) = p(N)$.*

(b) *$p(L) = p(AL)$ for every nilpotent minimal left S-ideal L, and $p(N) = p(AN)$.*

(c) *$L \subseteq LA$ for every nilpotent minimal left S-ideal L.*

Proof. Suppose that A possesses an identity element. That the condition (a) is satisfied has been observed already, while (b) and (c) follow at once.

We shall complete the proof by showing that (b) implies (a), and that each of the conditions (a) and (c) implies the existence of a right identity; the existence of an identity follows from Lemma 3, since we are continuing under the assumptions made concerning A and S at the beginning of §10.

Since we have assumed the descending chain condition for left S-ideals, A is a semi-primary S-ring unless it is nilpotent. In any case there exists an element $e = e^2$ in A such that

$$A = eA \oplus R_0 = Ae \oplus L_0$$

where R_0 and L_0 are respectively a left and a right S-ideal of A contained in N such that $eR_0 = L_0e = 0$. If $L_0 \neq 0$, we infer from the descending chain condition that there exists a minimal left S-ideal L of A such that $L \subseteq L_0$. Since L is minimal, $NL = 0$, i.e., $L \subseteq I_r(N)$. Moreover, $LA = LeA + LR_0 = LR_0 \subseteq LN$, so that $LA = LN$.

Assuming (a) we have $L \subseteq I_r(N) = p(N) = q(N^*) = q(N) \subseteq I_l(N)$. Hence $LN = 0$ which implies that $LA = 0$. Then $A \subseteq I_r(L) = p(L)$, hence $L = 0$. But this contradicts the assumption that $L_0 \neq 0$, proving that e is a right identity of A.

Next assume (b). For any subset X we have the implications $XI_r(X) = 0$, $(A, XI_r(X)) = 0$, $(AX, I_r(X)) = 0$, $I_r(X) \subseteq p(AX)$. Hence if $p(X) = p(AX)$, $I_r(X) = p(X)$. Thus (*b*) implies (a).

Finally assume (c). Then we have the following implications: $NL = 0$, $(NL, A) = 0$, $(N, LA) = 0$, $L \subseteq LA \subseteq p(N) = q(N^*) = q(N) \subseteq I_l(N)$, $LN = 0$, $LA = 0$, $L = 0$. This again contradicts the assumption that $L_0 \neq 0$, proving that e is a right identity, and completing the proof of the proposition.

III. Applications

In this last part we mention some applications of the general discussion of the preceding parts, in particular relating our results to certain known results.

11. Algebras. For a first application of our general theory we consider a finite dimensional algebra A over a field F, assuming that A has an identity element e, so that F can be identified with a subring of the center of A.

A module M is called a *unitary representation module* for A if M is a (right) A-module such that $ue = u$ for all u in M, and has finite dimension over F. Then M corresponds to a representation Γ of A by matrices in F such that $\Gamma(e)$ is the identity matrix, i.e., to a *unitary representation* Γ *of* A.

We can apply the discussion of Parts I and II, reading "unitary representation module" for "A,F-module", and "finite vector module" for "F-bimodule". Note that a simplification takes place because of the fact that, if M is a subrepresentation module of a unitary representation module H of A (in which case Γ is called a top constituent of the representation Δ corresponding to H) then M is automatically an A,F-retract of H since there exists a vector submodule of H complementary to M. If M is a quotient representation module of H (in which case Γ is called a bottom constituent of Δ) then M is an A,F-protract of H.

If a unitary representation module M has dimension m over F then it decomposes as a vector module into a direct sum of m copies of F

$$M \simeq m \times F.$$

Hence, by §7 and formulas (4) and (4′) of Part I we have

$$(1) \qquad I_{A,S}(M) \simeq {}_Am \times I_{A,S}(F) \simeq {}_Am \times (F \otimes_S A) \simeq {}_Am \times A$$

$$(1') \qquad P_{A,S}(M) \simeq {}_Am \times S\text{-Hom}_r(A,S) = m \times A'.$$

In (1), A is to be considered as the A-module corresponding to the first regular representation of A. In (1′), A is to be considered as the left A-module corresponding to the regular anti-representation of A, so that $A' = S\text{-Hom}_r(A,S)$ corresponds to the second regular representation of A.

M corresponds to a unitary representation Γ of degree m.

By Theorems 3, 3′ we have

THEOREM 7. *A unitary representation* Γ *of* A *of degree* m *is a bottom constituent of* $m \times$ *the first regular representation of* A *and a top constituent of* $m \times$ *the second regular representation of* A.

Let us call Γ *projective* or *injective* according as M is A,F-projective or A,S-injective, in the sense of 4.

If M happens to be indecomposable, then by the Remak theorem and (1), M is isomorphic with a direct summand of $I(M)$ if and only if it is isomorphic with a direct summand of A. Hence by Theorem 4 we have

THEOREM 8. *If Γ is an indecomposable unitary representation of A in F, the following conditions imply each other:*

(a) *Γ is a principal indecomposable representation of A, i.e., Γ is equivalent to an indecomposable component of the first regular representation of A.*

(b) *Γ is projective.*

(c) *whenever Γ is a bottom constituent of a unitary representation Δ of A, Γ is equivalent to a component of Δ.*

The dual of Theorem 8 corresponding to Theorem 4′ is obtained by interchanging "first" with "second" regular representation, "projective" with "injective", and "bottom" with "top" constituent.

COROLLARY. *A unitary representation Γ is projective if and only if it is a direct sum of principal indecomposable representations, while Γ is injective if and only if it is a direct sum of indecomposable components of the second regular representation.*

An algebra A over a field F is a self-dual F-ring in the sense of Part II if and only if its two regular representations are equivalent (Proposition 1). Thus the algebras with identity elements which are self-dual with respect to the ground field are precisely the Frobenius algebras **(13)**. Lemma 3 and Proposition 3 provide necessary and sufficient conditions for an algebra A having its two regular representations equivalent to possess an identity element.

By Theorem 6 and Theorem 8 and its dual we have

THEOREM 9. *Let A be an algebra over a field F, and assume that A possesses an identity element. Let Γ be an indecomposable unitary representation of A in F of degree m. Let $a_1, \ldots, a_n$ be a basis of A, and assume that there exists a set $\bar{a}_1, \ldots, \bar{a}_n$ of elements of A satisfying* (9) *of Part II. Then the condition*

(**) *there exists a matrix X in F of degree m such that*

$$\sum_{i=1}^{n} \Gamma(\bar{a}_i) X \, \Gamma(a_i) = I_m$$

(where I_m is the identity matrix of degree m)

implies that Γ is a component of both the first and second regular representations of A.

If A is a Frobenius algebra, and $a_1, \ldots, a_n$ and $\bar{a}_1, \ldots, \bar{a}_n$ are dual bases of A, then (**) *characterizes the direct sums of principal indecomposable representations of A.*

12. Separable algebras. We shall also record the following characterization of separable algebras. The equivalence of (b) of our theorem with separability was proved by Hochschild **(10**, Theorem 5**)**; that condition (b) below is equivalent with Hochschild's condition (ii) is fairly obvious.

THEOREM 10. *Each of the following conditions is necessary and sufficient for an algebra A with identity element e to be separable:*

(a) *A is a Frobenius algebra, and there exists an element a in A such that $\sum \bar{a}_i a a_i = e$, where $a_1, \ldots, a_n$ and $\bar{a}_1, \ldots, \bar{a}_n$ are dual bases of A.*

(b) *to each basis $a_1, \ldots, a_n$ of A there corresponds a set $\bar{a}_1, \ldots, \bar{a}_n$ of elements of A satisfying condition* (9) *of Part II such that $\sum \bar{a}_i a_i = e$.*

Proof. As is well known, a separable algebra is Frobenius, and even symmetric, for it has an identity element, and its reduced trace induces an invariant bilinear form. Hence to prove that (a) is satisfied by separable algebras A we have to show the existence of an element a in A such that $\sum \bar{a}_i a a_i = e$. By a standard device, this question is reduced to the case where A is a direct sum of full matrix rings in F,

$$A = \sum_{\alpha=1}^{k} {}_{\alpha}F_{n_\alpha}.$$

Let E^{α}_{ij} $(\alpha = 1, \ldots, k;\ i, j = 1, \ldots, n_\alpha)$ be the matrix units. Then $(E^{\alpha}_{ij}, E^{\beta}_{kl}) = \delta_{\alpha\beta}\delta_{il}\delta_{jk}$ determines an invariant F-bilinear form on A such that the basis of A dual to the basis consisting of the E^{α}_{ij} is defined by

$$\overline{E^{\alpha}_{ij}} = E^{\alpha}_{ji}.$$

Let $T = \sum E^{\alpha}_{ii}$ so that $TE^{\alpha}_{ij} = 0$ if $i \neq 1$, $TE^{\alpha}_{ij} = E^{\alpha}_{ij}$. Then

$$\sum_{\alpha, i, j} \overline{E^{\alpha}_{ij}} T E^{\alpha}_{ij} = \sum_{i, \alpha} E^{\alpha}_{ii} = I$$

proving (a) in this case.

Assume (a) and write $\bar{a}_i = \bar{a}_i a$. Then $a_1, \ldots, a_n$ and $\bar{a}_1, \ldots, \bar{a}_n$ satisfy condition (9) of Part II since this is the case for $a_1, \ldots, a_n$ and $\bar{a}_1, \ldots, \bar{a}_n$ according to Proposition 3. Hence (a) implies (b).

By Theorem 6, condition (b) implies that every unitary representation module of A is a direct summand of every unitary representation module of which it is a representation submodule. Hence A is semi-simple. Since condition (b) survives any extension of the ground field, this implies that A is separable. Thus (a) and (b) are each equivalent to separability.

13. Groups. Let G be a group, and denote by A the ring of all linear combinations

$$\sum_{g \in G} a_g g,$$

where a_g is a rational integer, and almost all a_g are zero. Let S be a subring of A spanned by a subgroup H of finite index. A set L of representatives of the cosets xH is a left S-basis of A and the set L^{-1} of inverses of elements of L is a dual right S-basis. Therefore A is a self-dual S-ring in the sense of Part II. The generalized Casimir operators have in this case the form

$$\sum_{x \in L} x^{-1} \alpha x$$

(cf. §9), and application of Theorem 6 yields at once Theorems 1, 1′ of a previous note of Higman **(7)**, which includes a theorem of Eckmann **(2)** and coincides with Gaschütz's generalization **(5)** of the Maschke theorem in case $S = 1$.

Assume that G is finite, and let M be a representation module for G in a field of prime characteristic p. If the subgroup H contains a p-Sylow subgroup of G, multiplication by the index $G:H$ induces an automorphism of M, and taking α to be the inverse of this automorphism we have $\sum x^{-1}\alpha x = 1$. We thus obtain Theorem 2 of **(7)**. See **(8)** for another application.

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ON ORDERS IN SEPARABLE ALGEBRAS

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Introduction. The present note began from the observation that the arguments produced by J-M. Maranda in developing his very interesting theory of representations of groups by automorphisms of modules over Dedekind rings **(4, 5)** were applicable without essential change to arbitrary orders, instead of just group rings, provided that a suitable generalization of Theorem 1 of **(4)** could be supplied. We prove that when the ring of integers is a Dedekind ring a certain integral ideal $I(\mathfrak{G})$ vanishes if and only if $\mathfrak{G}$ is an order in a separable algebra, thus extending Maranda's results to these orders and indicating that an essential change can be expected in going beyond this case.

The author is indebted to Professor Maranda for the opportunity of studying **(5)** before its publication.

Notations. The following notations will be fixed throughout.

$\mathfrak{g}$ = Dedekind ring.
K = quotient field of $\mathfrak{g}$.
A = (finite dimensional linear associative) algebra over K with identity element e.
$\mathfrak{G}$ = $\mathfrak{g}$-order in A.

1. The ideal $I(\mathfrak{G})$. By a *two-sided G-module* we shall understand a module T having $\mathfrak{G}$ both as a ring of left and right operators such that

$$(\xi u)\eta = \xi(u\eta), \quad eu = ue = u \qquad (\xi, \eta \in G, u \in T),$$

which is finitely generated over $\mathfrak{g}$.

For such a two-sided $\mathfrak{G}$-module T we shall denote by $Z(T)$ the $\mathfrak{g}$-module of all $\mathfrak{g}$-homomorphisms ϕ of $\mathfrak{G}$ into T such that

$$\phi(\xi\eta) = \xi\phi(\eta) + \phi(\xi)\eta \qquad (\xi, \eta \in \mathfrak{G}), \tag{1}$$

and by $B(T)$ the submodule of $\phi \in Z(T)$ for which there exist elements $u \in T$ such that

$$\phi(\omega) = \omega u - u\omega \qquad (\omega \in \mathfrak{G}). \tag{2}$$

We shall use $H(T)$ to denote the quotient module $Z(T)/B(T)$.

In the language of cohomology theory $Z(T)$ and $B(T)$ are modules of 1-*dimensional cocycles* and *coboundaries* respectively, and $H(T)$ is the 1-*dimensional cohomology* group for T.

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A *right* $\mathfrak{G}$*-module*, or simply a $\mathfrak{G}$*-module* will be understood to be a module having $\mathfrak{G}$ as a ring of right operators such that e acts as the identity operator, which is finitely generated over $\mathfrak{g}$. If M and N are $\mathfrak{G}$-modules, the module $\mathrm{Hom}(M, N)$ of all $\mathfrak{g}$-homomorphisms of M into N can be turned into a two-sided $\mathfrak{G}$-module by defining

$$(f\omega)(u) = \omega(f(u)), \quad (\omega f)(u) = f(u\omega) \qquad (\omega \in \mathfrak{G}).$$

Taking $T = \mathrm{Hom}(M, N)/\mathfrak{a}\mathrm{Hom}(M, N)$, where $\mathfrak{a}$ is an integral ideal of $\mathfrak{g}$, we obtain the module $Z(T)$ of $\mathfrak{a}$-*modular binding systems* of M and N and the submodule of $\mathfrak{a}$-modular binding systems *strongly equivalent to* 0 **(4)**.

We shall be particularly concerned with the annihilators $I(T)$ of the $\mathfrak{g}$-modules $Z(T)$, and the intersection for all two-sided $\mathfrak{G}$-modules T of the ideals $I(T)$, which intersection we shall denote by $I(\mathfrak{G})$. From the above we see that a theorem to the effect that $I(\mathfrak{G}) \neq 0$ would be a "suitable" generalization of Maranda's Theorem 1 in **(4)**. In §3 we shall prove that $I(\mathfrak{G}) \neq 0$ if and only if A is separable. In §4 we shall show how to construct $I(\mathfrak{G})$ from an invariant bilinear form on A; in particular, if $\mathfrak{G}$ is the group ring of a finite group of order N then $I(\mathfrak{G})$ is the principal ideal generated by N.

2. Separable algebras. In order to obtain the results mentioned in §1 we shall make use of a characterization of separable algebras which we established as a corollary to general results in **(2)**. For the convenience of the reader we include a direct proof here.

Assume that A is an algebra over a field K with identity element e. Assume furthermore that there exists an *invariant* bilinear form f on A, that is, f is a non-singular bilinear form defined on A with values in K such that

$$f(xy, z) = f(x, yz) \qquad (x, y, z \in A). \tag{3}$$

This is equivalent to the assumption that A is a *Frobenius algebra* **(2)**.

If g is a second invariant bilinear form on A, then

$$g(x, y) = f(x, yc) \tag{4}$$

with c a unit of A **(2)**.

Let $a_1, \ldots, a_n$ be a basis of A over K, then

$$f(a_i, \hat{a}_j) = \delta_{ij} = f(\hat{a}_j, a_i) \tag{5}$$

defines two dual bases $\hat{a}_1, \ldots, \hat{a}_n$ and $\hat{a}_1, \ldots, \hat{a}_n$ of A which coincide if and only if A is symmetric. Simple computation using (3) shows that, for $a \in A$,

$$a_i a = \sum s_{ij} a_j,\ s_{ij} \in K, \text{ if and only if } a\hat{a}_j = \sum \hat{a}_i s_{ji}, \tag{6}$$

while

$$aa_i = \sum a_j t_{ji},\ t_{ji} \in K, \text{ if and only if } \hat{a}_i a = \sum t_{ij} \hat{a}_j. \tag{6'}$$

Using (5), (6) and (6′) we verify that, for $a \in A$,

$$c_f(a) = \sum \hat{a}_i a a_i = \sum a_i a \hat{a}_i$$

is an element of the center Z of A, independent of the choice of basis $a_1, \ldots, a_n$, and that c_f is a Z-homomorphism of A into Z. In particular $c_f(A)$ is an ideal of Z. c_f can be referred to as the *Casimir operator* determined by f **(2)**.

If g is an invariant bilinear form related to f by (4) then clearly

$$c_f(a) = c_g(ca) \qquad (a \in A).$$

Hence $c_f(A)$ is independent of the choice of invariant bilinear form f, and we can write $c_f(A) = c(A)$.

THEOREM 1. *Each of the following conditions is necessary and sufficient for an algebra A with identity element e to be separable.*

(i) *A is a Frobenius algebra such that $c(A)$ = the center of A.*

(ii) *Corresponding to a basis $a_1, \ldots, a_n$ of A there is a set of elements $\bar{a}_1, \ldots, \bar{a}_n$ of A such that*

$$\sum \bar{a}_i a_i = e \tag{7}$$

and such that, for $a \in A$,

$$a_i a = \sum s_{ij} a_j, \ s_{ij} \in K, \text{ implies } a\bar{a}_i = \sum \bar{a}_j s_{ji}. \tag{8}$$

We note that some time ago Hochschild **(3**, Theorem 5**)** established the equivalence with separability of a slightly different form of Condition (ii).

Proof of Theorem 1. *Separability implies* (i): If A is separable, there exists a finite extension E of K such that A_E is isomorphic with a direct sum of full matrix rings over E,

$$A_E \simeq \sum{}^{\circ} E_{n(\alpha)}.$$

For a basis of A_E over E choose the matrix units e^{α}_{ij} of the $E_{n(\alpha)}$. Then

$$f^*(e^{\alpha}_{ij}, e^{\beta}_{kl}) = \delta_{\alpha\beta}\delta_{il}\delta_{jk}$$

defines a symmetric invariant bilinear form f^* on A_E. In fact $f^*(x, y) = S(xy)$, where S is the reduced trace on A_E **(1**, p. 33**)**. For the dual basis of A_E determined by f^* we have

$$\overline{e^{\alpha}_{ij}} = e^{\alpha}_{ji}.$$

Setting $b^* = \sum e^{\alpha}_{ii}$ we can verify that

$$c_{f^*}(b^*) = \sum \overline{e^{\alpha}_{ij}} b^* e^{\alpha}_{ij} = I.$$

Now f^* induces an invariant bilinear form f on A. The existence of an element $b \in A$ such that $c_f(b) = e$ is seen to be equivalent to the existence in K of a solution of a system of linear equations with coefficients in K. What we have proved above implies that these equations have a solution in E, hence they already have a solution in K, proving that such an element $b \in A$ exists. Since $c_f(A)$ is an ideal of Z, this proves that $c(A) = c_f(A) = Z$.

It is clear that (ii) is a consequence of (i).

(ii) *implies separability.* Let Ξ be a representation of A in an extension F of K (with $\Xi(e) = I$), and consider a reduction

$$\Xi \backsim \begin{bmatrix} \Gamma & \phi \\ & \Delta \end{bmatrix}$$

of Ξ. Here Γ and Δ are representations of A in F and ϕ is a linear mapping of A into the vector module over K of all $n \times m$ matrices with coefficients in F, where m, n are respectively the degrees of Γ and Δ, and ϕ satisfies the identity

$$\phi(xy) = \Gamma(x)\,\phi(y) + \phi(x)\,\Delta(y) \qquad (x, y \in A). \tag{9}$$

Let $a_1, \ldots, a_n$ be a basis of A, and assume the existence of a set of elements $\bar{a}_1, \ldots, \bar{a}_n$ as in (ii). From (9) we have

$$\Gamma(a_i)\,\phi(a_j) = \phi(a_i a_j) - \phi(a_i)\,\Delta(a_j). \tag{10}$$

Multiplying (10) on the left by $\Gamma(\bar{a}_i)$, summing over i, and making use of (7) and (8) we obtain $\phi(a_j) = \Gamma(a_j)\,T - T\Delta(a_j)$, $(j = 1, \ldots, n)$, where $T = \sum \Gamma(\bar{a}_i)\,\phi(a_i)$. It follows that

$$\Xi \backsim \begin{bmatrix} I & T \\ & I \end{bmatrix}\begin{bmatrix} \Gamma & \phi \\ & \Delta \end{bmatrix}\begin{bmatrix} I & -T \\ & I \end{bmatrix} = \begin{bmatrix} \Gamma & 0 \\ & \Delta \end{bmatrix}$$

proving that Ξ is completely reducible, and consequently that A is separable

3. The generic cocycle. In order to study the ideal $I(\mathfrak{G})$ of $\mathfrak{g}$ defined in §1 we adapt a method of Hochschild **(3, §4)**. By $P = P(\mathfrak{G})$ we denote the product $\mathfrak{G} \otimes_{\mathfrak{g}} \mathfrak{G}$, with operators defined as follows

$$(\omega\zeta \otimes \eta) = \omega\zeta \otimes \eta$$

$$(\zeta \otimes \eta)\,\omega = (\zeta \otimes \eta\omega) - (\zeta\eta \otimes \omega) \qquad (\omega \in \mathfrak{G}).$$

P is a two-sided $\mathfrak{G}$-module as we are using the term except for the fact that e does not act as the identity operator on the right. But we may of course define $Z(P)$, $B(P)$, $H(P)$ and $I(P)$ as for ordinary two-sided $\mathfrak{G}$-modules. The *generic* 1*-cocycle* is the element $F \in Z(P)$ defined by

$$F(\omega) = e \otimes \omega \qquad (\omega \in \mathfrak{G}).$$

LEMMA 1. *$I(G) = I(P) = I(F)$, where $I(F) = \{\lambda \in g | \lambda F \in B(P)\}$.*

Proof. Clearly $I(P) \subseteq I(F)$. If now T is any two-sided $\mathfrak{G}$-module and $f \in Z(T)$, the $\mathfrak{g}$-homomorphism μ_f of P into T defined by

$$\mu_f\colon (\zeta \otimes \eta) = \zeta \cdot f(\eta)$$

is seen to commute with the operators of $\mathfrak{G}$ on the left and on the right, and furthermore $F\mu_f = f$. Consequently for $\lambda \in \mathfrak{g}$, $\lambda F \in B(P)$ implies $\lambda f \in B(T)$, that is, $I(F) \subseteq I(T)$. Since this holds for all T, $I(F) \subseteq I(\mathfrak{G})$. On the other

hand, $P' = Pe$ is an ordinary two-sided $\mathfrak{G}$-module so that $I(\mathfrak{G}) \subseteq I(P')$, and, since $H(P) \simeq H(P')$, $I(P) = I(P')$. We have

$$I(P) \subseteq I(F) \subseteq I(\mathfrak{G}) \subseteq I(P)$$

proving the lemma.

Using this characterization of $I(\mathfrak{G})$ it is straightforward to verify

LEMMA 2. *If $\mathfrak{o} = K$ or $\mathfrak{o}$ = the ring of all elements of K regular with respect to a given finite set of prime ideals of $\mathfrak{o}$ then $I(\mathfrak{o}\mathfrak{G}) = \mathfrak{o}I(\mathfrak{G})$.*

Moreover we can prove

LEMMA 3. *If $\mathfrak{G}$ has a linearly independent $\mathfrak{g}$-basis $\omega_1, \ldots, \omega_n$, then $I(\mathfrak{G})$ = the totality of elements $\lambda \in \mathfrak{g}$ which can be written in the form $\lambda = \sum \tilde{\omega}_i\omega_i$ where the elements $\tilde{\omega}_i \in \mathfrak{G}$ are such that for $\omega \in \mathfrak{G}$*

$$\omega_i\omega = \sum \mu_{ij}\omega_j, \quad \omega_j \in K \text{ implies } \omega\tilde{\omega}_i = \sum \tilde{\omega}_j\mu_{ji}. \tag{11}$$

Proof. By Lemma 1, $\lambda \in I(\mathfrak{G})$ if and only if $\lambda F \in B(P)$, that is, if and only if there exists an element $u \in P$ such that $\lambda F(\omega) = \omega u - u\omega$ for all $\omega \in \mathfrak{G}$. We can write u uniquely in the form $u = \sum \tilde{\omega}_i \otimes \omega_i$, $\tilde{\omega}_i \in \mathfrak{G}$, and hence

$$\lambda \otimes \omega = \sum \omega\tilde{\omega}_i \otimes \omega_i - \sum \tilde{\omega}_i \otimes \omega_i\omega + \sum \tilde{\omega}_i\omega_i \otimes \omega. \tag{12}$$

Putting $\omega = e$ in (12) we obtain $\lambda = \sum \tilde{\omega}_i\omega_i$, hence (12) reduces to

$$\sum \omega\tilde{\omega}_i \otimes \omega_i = \sum \tilde{\omega}_i \otimes \omega_i\omega$$

from which we readily deduce (11).

Now we obtain the theorem which is our generalization of Maranda's Theorem 1 in **(4)**, namely

THEOREM 2. *$I(\mathfrak{G}) \neq 0$ if and only if A is separable.*

Proof. Taking $\mathfrak{G} = A$ in Lemma 3 and applying (ii) of Theorem 1 we have that A is separable if and only if $I(A) = K$. But by Lemma 2, $I(A) = KI(\mathfrak{G})$, so that the theorem follows.

The reader of Maranda's papers **(4)** and **(5)** will now see that it is only necessary to replace the ideal generated by the group order N by the ideal $I(\mathfrak{G})$ in order to carry over Maranda's results to orders in separable algebras.

4. Construction of $I(\mathfrak{G})$. Let us assume that there exists an invariant bilinear form f on A. In order to construct $I(\mathfrak{G})$ we proceed as follows. We denote by $\mathfrak{G}_f$ the *inverse different* corresponding to f,

$$\mathfrak{G}_f = \{a \in A | f(\mathfrak{G}, a) \subseteq \mathfrak{g}\}. \tag{13}$$

The *different* $\mathfrak{I}_f(G)$ is defined by

$$\mathfrak{I}_f(\mathfrak{G}) = \{a \in A | \mathfrak{G}_f a \subseteq \mathfrak{G}\}. \tag{14}$$

Notice that in case $\mathfrak{G}$ has a linearly independent $\mathfrak{g}$-basis $\omega_1, \ldots, \omega_n$, and $f(\omega_i, \bar{\omega}_j) = \delta_{ij}$, the elements $\bar{\omega}_1, \ldots, \bar{\omega}_n$ constitute a $\mathfrak{g}$-basis of $\mathfrak{G}_f$, so that $a \in \mathscr{I}_f(\mathfrak{G})$ if and only if $\bar{\omega}_i a \in \mathfrak{G}$ $(i = 1, \ldots, n)$.

If now $\mathfrak{o} = K$ or $\mathfrak{o}$ = the ring of all elements of K regular with respect to a finite set of prime ideals of $\mathfrak{o}$ we can readily verify that $(\mathfrak{o}\mathfrak{G})_f = \mathfrak{o}\mathfrak{G}_f$, and hence

LEMMA 4. $\mathscr{I}_f(\mathfrak{o}\mathfrak{G}) = \mathfrak{o}\mathscr{I}_f(\mathfrak{G})$.

We remark that (13) and (14) are not symmetric. Using our assumption that $\mathfrak{g}$ is a Dedekind ring, it is easy to prove that when f is *symmetric* the four possible definitions obtained by interchanging left and right in (13) and (14) produce the same $\mathscr{I}_f(\mathfrak{G})$.

Now we obtain $I(\mathfrak{G})$, namely, putting

$$D_f(\mathfrak{G}) = c_f(\mathscr{I}_f(\mathfrak{G}))$$

we can prove

THEOREM 3. *If f is an invariant bilinear form on A then*

$$I(\mathfrak{G}) = D_f(\mathfrak{G}) \cap \mathfrak{g}.$$

LEMMA 5. $D_f(\mathfrak{G}) \cap \mathfrak{g} \neq 0$ *if and only if A is separable.*

Proof. By (i) of Theorem 1 and the definition of D_f, A is separable if and only if $D_f(A) = Z$, i.e., $D_f(A) \cap K = K$. By Lemma 4, $D_f(A) \cap K = K[D_f(\mathfrak{G}) \cap \mathfrak{g}]$, so that the lemma follows.

Proof of Theorem 3. We proceed first under the assumption that G has a linearly independent g-basis $\omega_1, \ldots, \omega_n$. In this case it is clear from the definition of D_f and from Lemma 3 that $D_f(\mathfrak{G}) \cap \mathfrak{g} \subseteq I(\mathfrak{G})$.

If on the other hand $\lambda \in I(\mathfrak{G})$, then $\lambda = \sum \bar{\omega}_i \omega_i$ as in Lemma 3. Consider the linear endomorphism σ of A defined by $\sigma\colon \bar{\omega}_i \otimes \bar{\omega}_i$, where $f(\omega_i, \bar{\omega}_j) = \delta_{ij}$. For $\omega \in \mathfrak{G}$, $\omega_i \omega = \sum \mu_{ij} \omega_j$ with $\mu_{ij} \in g$. Then by (6), $\omega \bar{\omega}_i = \sum \bar{\omega}_j \mu_{ji}$, hence by (11), $\sigma\colon \omega \bar{\omega}_i \to \sum \bar{\omega}_j \mu_{ji} = \omega \bar{\omega}_i = \omega(\bar{\omega}_i \sigma)$, proving that σ is a $\mathfrak{G}$-endomorphism, and hence an A-endomorphism, of A considered as a left A-module. Thus σ is effected by right multiplication by an element $c \in A$, and $\bar{\omega}_i = \bar{\omega}_i c \in G$ so that $c \in \mathscr{I}^{-1}{}_f(\mathfrak{G})$. Hence $\lambda = \sum \bar{\omega}_i \omega_i = \sum \bar{\omega}_i c\, \omega_i \in D_f(\mathfrak{G})$, proving that $I(\mathfrak{G}) \subseteq D_f(\mathfrak{G})$. The desired equality is now proved in this case.

We now drop the assumption of the existence of a linearly independent $\mathfrak{g}$-basis for $\mathfrak{G}$. By Theorem 2 and Lemma 5, $I(\mathfrak{G}) = D_f(\mathfrak{G}) \cap \mathfrak{g} = 0$ if A is not separable, while if A is separable, $I(\mathfrak{G}) \neq 0$ and $D_f(\mathfrak{G}) \cap \mathfrak{g} \neq 0$. In the latter case let us denote by $\mathfrak{o}$ the ring of all elements of K regular with respect to all the prime divisors of $I(\mathfrak{G})$ and $D_f(\mathfrak{G}) \cap \mathfrak{g}$. Since $\mathfrak{o}$ is a principal ideal domain the $\mathfrak{o}$-order has a linearly independent $\mathfrak{o}$-basis. Hence, by Lemmas 2 and 4 and the case of our theorem proved above

$$\mathfrak{o}I(\mathfrak{G}) = I(\mathfrak{o}\mathfrak{G}) = D_f(\mathfrak{o}\mathfrak{G}) \cap \mathfrak{o} = \mathfrak{o}[D_f(\mathfrak{G}) \cap \mathfrak{g}].$$

It follows from the choice of $\mathfrak{o}$ that $I(\mathfrak{G}) = D_f(\mathfrak{G}) \cap \mathfrak{g}$, completing the proof of our theorem.

For an example, let us suppose that $\mathfrak{G}$ is the group ring of a finite group of order N. An invariant bilinear form f on A is defined by

$$f(g, h^{-1}) = \delta_{gh} \qquad (g, h \text{ group elements}).$$

Then $\mathfrak{G}_f = \mathfrak{G}$, $I(\mathfrak{G}) = D_f(\mathfrak{G}) \cap \mathfrak{g} = c_f(\mathfrak{G}) \cap \mathfrak{g} = N\mathfrak{g}$.

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ON THE EQUIVALENCE OF REPRESENTATIONS OF FINITE GROUPS BY GROUPS OF AUTOMORPHISMS OF MODULES OVER DEDEKIND RINGS

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1. Introduction. Let $\mathfrak{i}$ be a Dedekind ring whose quotient field we denote by K. If $\mathfrak{p}$ is a prime ideal of $\mathfrak{i}$, let $\mathfrak{o}$ denote the ring of all $\mathfrak{p}$-regular elements of K. If $\mathfrak{M}$ is a torsion free $\mathfrak{i}$-module, let $K\mathfrak{M}$ denote the smallest K-module into which $\mathfrak{M}$ can be embedded and let $\mathfrak{o}\mathfrak{M}$ denote the $\mathfrak{o}$-submodule of $K\mathfrak{M}$ generated by $\mathfrak{M}$.

Let $\mathfrak{G}$ be a group of finite order N and let A, $\mathfrak{O}$ and $\mathfrak{J}$ denote the group rings of $\mathfrak{G}$ over K, $\mathfrak{o}$ and $\mathfrak{i}$ respectively.

If $\mathfrak{M}$ is an $\mathfrak{J}$-module which is torsion free over $\mathfrak{i}$, then there is a unique way of defining $K\mathfrak{M}$ ($\mathfrak{o}\mathfrak{M}$) as an A-module ($\mathfrak{O}$-module) so that it is an extension of the $\mathfrak{J}$-module $\mathfrak{M}$. If $\mathfrak{M}$ and $\mathfrak{N}$ are two $\mathfrak{J}$-modules that are torsion free over $\mathfrak{i}$, we will say that $\mathfrak{M}$ and $\mathfrak{N}$ are

(1) $\mathfrak{i}$-equivalent if they are $\mathfrak{J}$-isomorphic;

(2) K-equivalent if $K\mathfrak{M}$ and $K\mathfrak{N}$ are A-isomorphic;

(3) $\mathfrak{o}$-equivalent if $\mathfrak{o}\mathfrak{M}$ and $\mathfrak{o}\mathfrak{N}$ are $\mathfrak{O}$-isomorphic;

(4) of the same genus if they are $\mathfrak{o}$-equivalent for every prime ideal $\mathfrak{p}$ of $\mathfrak{i}$.

Let $\mathfrak{S}$ be a class of K-equivalent $\mathfrak{J}$-modules that are torsion free and finitely generated over $\mathfrak{i}$. We will assume that $\mathfrak{S}$ is complete in the sense that if an $\mathfrak{J}$-module $\mathfrak{M}$ is K-equivalent to an $\mathfrak{J}$-module in $\mathfrak{S}$, then $\mathfrak{M}$ is $\mathfrak{i}$-equivalent to an $\mathfrak{J}$-module in $\mathfrak{S}$. Let $\mathfrak{R}$, $\mathfrak{R}(\mathfrak{p})$ and $\mathfrak{R}_g$ denote the partitions of $\mathfrak{S}$ into complete subclasses of $\mathfrak{i}$-equivalent $\mathfrak{J}$-modules, $\mathfrak{o}$-equivalent $\mathfrak{J}$-modules and $\mathfrak{J}$-modules of the same genus respectively, and let $\mathbf{r}$, $\mathbf{r}(\mathfrak{p})$ and $\mathbf{r}_g$ denote the number of subclasses of $\mathfrak{S}$ determined by $\mathfrak{R}$, $\mathfrak{R}(\mathfrak{p})$ and $\mathfrak{R}_g$ respectively.

We will show first of all that if $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ are the prime ideal divisors of N, then

$$\mathfrak{R}_g = \bigcap_{i=1}^{r} \mathfrak{R}(\mathfrak{p}_i)$$

and that

$$\mathbf{r}_g = \prod_{i=1}^{r} \mathbf{r}(\mathfrak{p}_i).$$

Secondly, we will show that if for every $\mathfrak{J}$-module $\mathfrak{M} \in \mathfrak{S}$, the representation of A belonging to $K\mathfrak{M}$ is abolutely irreducible, then $\mathbf{r} = h\mathbf{r}_g$, where h is the ideal class number of $\mathfrak{i}$. In the case where $\mathfrak{i}$ is the ring of algebraic integers of

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an algebraic number field K, then h is finite, and we will show that each $\mathfrak{r}(\mathfrak{p}_i)$ is finite, so that $\mathfrak{r}$ is also finite.

In §2, we will speak in terms of matrix representations of $\mathfrak{G}$. This will not constitute a restriction, since in this section we will speak of representations of $\mathfrak{G}$ over principal ideal rings.

2. Integral representations of finite groups over fields with $\mathfrak{P}$-adic valuations. Let $\mathfrak{G}$ be a group of finite order N and let $\mathfrak{o}$ be a commutative ring with unity element. By an $\mathfrak{o}$-representation of $\mathfrak{G}$ of degree n, we mean a homomorphism of Γ of $\mathfrak{G}$ into the multiplicative semi-group of all $n \times n$ $\mathfrak{o}$-matrices which maps the identity element of $\mathfrak{G}$ onto the identity matrix. We will denote the image by Γ of any $x \in \mathfrak{G}$ by Γ_x.

If R is a commutative extension ring of $\mathfrak{o}$, then two $\mathfrak{o}$-representations Γ and Δ of $\mathfrak{G}$ will be said to be R-equivalent if they are of the same degree n and if there is an invertible $n \times n$ R-matrix T such that $\Gamma_x T - T\Delta_x = 0$ for all $x \in \mathfrak{G}$. If the $\mathfrak{o}$-representation Γ of $\mathfrak{G}$ is $\mathfrak{o}$-equivalent to a representation of the type

$$x \to \begin{pmatrix} \Gamma'_x & \Lambda_x \\ 0 & \Gamma''_x \end{pmatrix}$$

where Γ' and Γ'' are $\mathfrak{o}$-representations of $\mathfrak{G}$, then we say that Γ is reducible; otherwise, we say that Γ is irreducible.

Let K be a field of characteristic zero with a $\mathfrak{P}$-adic valuation ϕ, that is, ϕ is non-archimedean and discrete of rank 1, and the residue class field determined by ϕ is of finite characteristic. If K is complete with respect to ϕ, then K is said to be a $\mathfrak{P}$-adic field. Let $\mathfrak{o}$ denote the ring of integers of K, i.e., the set of all elements with values $\leqslant 1$, let $\mathfrak{P}$ denote the prime ideal of $\mathfrak{o}$, i.e., the set of all elements with values <1, and let p be a generator of $\mathfrak{P}$. In general, we will denote the complete extension of K with respect to ϕ by K^*. The ring of integers of K^* will be denoted by $\mathfrak{o}^*$ and its prime ideal by $\mathfrak{P}^*$. Let $\mathfrak{P}^{k_0}$ be the highest power of $\mathfrak{P}$ dividing N.

For completeness, we will state some theorems obtained previously by the author, and refer the reader to **(2)** for the proofs.

THEOREM 1. *If Γ and Δ are two $\mathfrak{o}$-representations of $\mathfrak{G}$, then Γ and Δ are $\mathfrak{o}$-equivalent if and only if the $\mathfrak{P}^k$-modular representations $\bar{\Gamma}$ and $\bar{\Delta}$, induced by Γ and Δ respectively, are $(\mathfrak{o}/\mathfrak{P}^k)$-equivalent, for any $k > k_0$.*

For the proof, we refer the reader to **(2**, p. 347**)**. Although in the paper referred to one is only concerned with the case where K is complete, it is obvious that this condition is not essential for this theorem. The argument applied is due to Schur **(3)**.

COROLLARY 1. *Γ and Δ are $\mathfrak{o}^*$-equivalent if and only if they are $\mathfrak{o}$-equivalent.*

Proof. Assume that Γ and Δ are $\mathfrak{o}^*$-equivalent, i.e., that Γ and Δ are of the same degree n and that there is an $n \times n$ $\mathfrak{o}^*$-matrix T such that $\Gamma_x T - T\Delta_x = 0$

for all $x \in \mathfrak{G}$, and that $|T| \notin \mathfrak{P}^*$. Let T' be an $\mathfrak{o}$-matrix congruent to T, modulo $\mathfrak{P}^{*k}$. Such a matrix always exists since $(\mathfrak{o}/\mathfrak{P}^k) \cong (\mathfrak{o}^*/\mathfrak{P}^{*k})$. Then

$$\Gamma_x T' - T'\Delta_x \equiv 0 \pmod{\mathfrak{P}^k}$$

for all $x \in \mathfrak{G}$ and $|T'| \notin \mathfrak{P}$. Taking $k > k_0$, by Theorem 1 Γ and Δ are $\mathfrak{o}$-equivalent.

COROLLARY 2. *If $(\mathfrak{o}/\mathfrak{P})$ is a finite ring, then any class of K-equivalent $\mathfrak{o}$-representations of $\mathfrak{G}$ contains only a finite number of complete subclasses of $\mathfrak{o}$-equivalent representations.*

Proof. From Theorem 1, the number of classes of $\mathfrak{o}$-equivalent $\mathfrak{o}$-representations of $\mathfrak{G}$ of a given degree is less than or equal to the number of classes of $(\mathfrak{o}/\mathfrak{P}^k)$-equivalent $(\mathfrak{o}/\mathfrak{P}^k)$-representations of $\mathfrak{G}$ of the same degree, for any $k > k_0$, and if $(\mathfrak{o}/\mathfrak{P}^k)$ is a finite ring, this number is evidently finite.

THEOREM 2. *If $N \notin \mathfrak{P}$, and if the $\mathfrak{o}$-representation Γ of G is $\mathfrak{o}$-equivalent to a representation of the type*

$$x \to \begin{pmatrix} \Gamma'_x & \Lambda_x \\ 0 & \Gamma''_x \end{pmatrix},$$

where Γ' and Γ'' are $\mathfrak{o}$-representations of $\mathfrak{G}$, then Γ is $\mathfrak{o}$-equivalent to the representation

$$x \to \begin{pmatrix} \Gamma'_x & 0 \\ 0 & \Gamma''_x \end{pmatrix}.$$

For the proof, see **(2**, p. 353**)**. The remarks made after Theorem 1 also apply to this theorem.

THEOREM 3. *If K is complete, if $\mathfrak{D}$ is an $\mathfrak{o}$-representation of $\mathfrak{G}$ and if $\mathfrak{D}$ is $\mathfrak{o}$-equivalent to a representation of the type*

$$x \to \begin{pmatrix} \Gamma_x & \Lambda_x \\ p^k\theta_x & \Delta_x \end{pmatrix}$$

where Γ and Δ induce $\mathfrak{P}^k$-modular representations of $\mathfrak{G}$ and where $k > 2k_0$, then $\mathfrak{D}$ is $\mathfrak{o}$-equivalent to a representation of the type

$$x \to \begin{pmatrix} \Gamma^*_x & \Lambda_x \\ 0 & \Delta^*_x \end{pmatrix}$$

where Γ^* and Δ^* are $\mathfrak{o}$-representations of $\mathfrak{G}$ that coincide with Γ and Δ respectively, modulo $\mathfrak{P}^{k-k_0}$.

For the proof, see **(2**, p. 348**)**.

COROLLARY. *If K is complete, if $N \notin \mathfrak{P}$, if Γ and Δ are irreducible $\mathfrak{o}$-representations of $\mathfrak{G}$ of degree n and if T is a non-zero $n \times n$ $\mathfrak{o}$-matrix such that $\Gamma_x T - T\Delta_x = 0$ for all $x \in \mathfrak{G}$, then $T = p^m T'$, where T' is an $\mathfrak{o}$-matrix and $|T'| \notin \mathfrak{P}$.*

Proof. Since $N \notin \mathfrak{P}$, by Theorem 3, the $\mathfrak{P}$-modular representations $\bar{\Gamma}$ and $\bar{\Delta}$ induced by Γ and Δ respectively are irreducible. Let p^m be the highest power of p dividing all the entries of T and let $T = p^m T'$. Surely,

$$\Gamma_x T' - T'\Delta_x \equiv 0 \pmod{\mathfrak{P}}$$

so that by Schur's lemma, T' is non-singular, modulo $\mathfrak{P}$, since p does not divide all the entries of T', i.e., $|T'| \notin \mathfrak{P}$.

THEOREM 4. *If $N \notin \mathfrak{P}$, then two $\mathfrak{o}$-representations Γ and Δ of $\mathfrak{G}$ are K-equivalent if and only if they are $\mathfrak{o}$-equivalent.*

Proof. Let us assume that Γ and Δ are K-equivalent and let us first suppose that K is complete. If Δ is irreducible, by **(5**, p. 283**)** Γ is also irreducible, so that the statement follows from the Corollary to Theorem 3. If Δ is reducible, by Theorem 2 we may assume that

$$\Delta = \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix}$$

where Δ_1 and Δ_2 are $\mathfrak{o}$-representations of $\mathfrak{G}$. Since Γ is K-equivalent to Δ, by **(5**, p. 283**)** and by Theorem 2, Γ is $\mathfrak{o}$-equivalent to a representation of the type

$$\begin{pmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{pmatrix}$$

where Γ_1 and Γ_2 are $\mathfrak{o}$-representations of $\mathfrak{G}$ that are K-equivalent to Δ_1 and Δ_2 respectively. If we use induction with respect to the number of irreducible constituents of Δ, the K-equivalence implies the $\mathfrak{o}$-equivalence of Γ_1 and Γ_2 with Δ_1 and Δ_2 respectively, so that Γ and Δ are $\mathfrak{o}$-equivalent.

Now, if we consider the general case, where K is not necessarily complete if Γ and Δ are K-equivalent, then we have just seen that they are $\mathfrak{o}^*$-equivalent, so that by Corollary 1 to Theorem 1 they are $\mathfrak{o}$-equivalent.

COROLLARY. *If $N \notin \mathfrak{P}$ and if $\mathfrak{D}$ is any $\mathfrak{o}$-representation of $\mathfrak{G}$, then in any reduction of $\mathfrak{D}$ into irreducible constituents, these constituents are unique up to order and $\mathfrak{o}$-equivalence.*

Proof. Let

$$\begin{pmatrix} \Gamma_1 & & * \\ & \Gamma_2 \ldots & \\ 0 & & \Gamma_r \end{pmatrix} \text{ and } \begin{pmatrix} \Delta_1 & & * \\ & \Delta_2 \ldots & \\ 0 & & \Delta_s \end{pmatrix}$$

be two reductions of $\mathfrak{D}$ into irreducible constituents. Then, by **(5**, p. 283**)**, the Γ_i and Δ_i are also irreducible when considered as K-representations of $\mathfrak{G}$. Therefore $r = s$ and there is a permutation $i_1, i_2, \ldots, i_r$ of the indices $1, 2, \ldots, r$ such that, for $j = 1, 2, \ldots, r$, Γ_j is K-equivalent to Δ_{i_j}. Then by Theorem 4, Γ_j is $\mathfrak{o}$-equivalent to Δ_{i_j}.

3. On modules over Dedekind rings. Let $\mathfrak{i}$ be a Dedekind ring, so that $\mathfrak{i}$ is an integral domain and every non-null fractional ideal of $\mathfrak{i}$ is invertible, and let K denote the quotient field of $\mathfrak{i}$.

Whenever we consider an $\mathfrak{i}$-module $\mathfrak{M}$, we will always assume that $1.u = u$, for all $u \in \mathfrak{M}$. If $\mathfrak{M}$ is an $\mathfrak{i}$-module, the set of all $\alpha \in \mathfrak{i}$ for which $\alpha u = 0$ for all $u \in \mathfrak{M}$ is an ideal of $\mathfrak{i}$ which we will call the annihilator of $\mathfrak{M}$.

We define the order ideal[1] of a finitely generated $\mathfrak{i}$-module $\mathfrak{M}$ as follows: If $\mathfrak{M}$ contains an element without torsion, then its order ideal is null, while if $\mathfrak{M}$ is a torsion module, its annihilator $\mathfrak{a}$ is non-null, and since both chain conditions hold for the integral ideals of $\mathfrak{i}$ that contain $\mathfrak{a}$, $\mathfrak{M}$ has a composition series and its order ideal is the product of the annihilators of its composition factors.

If $\mathfrak{N}$ is an $\mathfrak{i}$-submodule of $\mathfrak{M}$, since the ascending chain condition holds for the integral ideals of $\mathfrak{i}$, $\mathfrak{N}$ is also finitely generated. We denote the order ideal of $\mathfrak{M}/\mathfrak{N}$ by $(\mathfrak{M}:\mathfrak{N})$. From the definition, it follows immediately that $(\mathfrak{M}:O) = (\mathfrak{M}:\mathfrak{N}) \cdot (\mathfrak{N}:O)$ and that $(\mathfrak{M}:O)$ is contained in the annihilator of $\mathfrak{M}$.

Let $\mathfrak{o}$ denote the ring of all elements of K that are regular with respect to each element of a finite set of prime ideals of $\mathfrak{i}$. The ring $\mathfrak{o}$ is a Dedekind ring, indeed a principal ideal domain, and its ideals are just those that are generated over $\mathfrak{o}$ by ideals of $\mathfrak{i}$.

If $\mathfrak{M}$ is a torsion free $\mathfrak{i}$-module, let $K\mathfrak{M}$ denote the smallest K-module into which $\mathfrak{M}$ can be embedded and let $\mathfrak{o}\mathfrak{M}$ denote the $\mathfrak{o}$-submodule of $K\mathfrak{M}$ generated by $\mathfrak{M}$. From the definition of the order ideal, one can verify that if $\mathfrak{N}$ is an $\mathfrak{i}$-submodule of $\mathfrak{M}$, then $(\mathfrak{o}\mathfrak{M}:\mathfrak{o}\mathfrak{N}) = \mathfrak{o}(\mathfrak{M}:\mathfrak{N})$ and $(K\mathfrak{M}:K\mathfrak{N}) = K(\mathfrak{M}:\mathfrak{N})$.

LEMMA 1. *If $\mathfrak{M}$ is a finitely generated torsion free $\mathfrak{i}$-module and if $\mathfrak{a}$ is an integral ideal of $\mathfrak{i}$, then $(\mathfrak{M}:\mathfrak{a}\mathfrak{M}) = \mathfrak{a}^m$, where m is the rank of $\mathfrak{M}$ (dimension of $K\mathfrak{M}$).*

Proof. If $\mathfrak{a} = 0$, then the statement is trivially true so that we may suppose that $\mathfrak{a} \neq 0$. Let us assume that $\mathfrak{o}$ is the ring of all elements of K that are regular with respect to each of the prime ideal divisors of $(\mathfrak{M}:\mathfrak{a}\mathfrak{M})$ and $\mathfrak{a}$.

Since $\mathfrak{o}$ is a principal ideal domain, $\mathfrak{o}\mathfrak{M}$ has a linearly independent basis $u_1, \ldots, u_m$ over $\mathfrak{o}$, so that

$$\mathfrak{o}\mathfrak{M} = \sum_{i=1}^{m} \mathfrak{o}u_i.$$

Then

$$(\mathfrak{o}\mathfrak{a}) \cdot (\mathfrak{o}\mathfrak{M}) = \sum_{i=1}^{m} (\mathfrak{o}\mathfrak{a})u_i$$

[1]For a general definition of the order ideal of a finitely generated module over a commutative ring with unity element, see (6, chap. III, §3 and §5). Using the fundamental theorem of Steinitz on modules over Dedekind rings (4, and also e.g. 1), one may show that this general definition is equivalent to the one given above in the case of modules over Dedekind rings.

and

$$\frac{\mathfrak{o}\mathfrak{M}}{(\mathfrak{o}\mathfrak{a}) \cdot (\mathfrak{o}\mathfrak{M})} \cong \sum_{i=1}^{m} \frac{\mathfrak{o}u_i}{(\mathfrak{o}\mathfrak{a})u_i} \qquad \text{(direct sum)}.$$

Therefore, $\mathfrak{o}(\mathfrak{M}:\mathfrak{a}\mathfrak{M}) = (\mathfrak{o}\mathfrak{M}:(\mathfrak{o}\mathfrak{a}) \cdot (\mathfrak{o}\mathfrak{M})) = (\mathfrak{o}\mathfrak{a})^m = \mathfrak{o}\mathfrak{a}^m$, and because of the choice of $\mathfrak{o}$, $(\mathfrak{M}:\mathfrak{a}\mathfrak{M}) = \mathfrak{a}^m$.

If $\mathfrak{M}$ and $\mathfrak{N}$ are torsion free $\mathfrak{i}$-modules, and if T is an $\mathfrak{i}$-homomorphism of $\mathfrak{M}$ into $\mathfrak{N}$, then T may be extended to a K-homomorphism of $K\mathfrak{M}$ into $K\mathfrak{N}$ in a unique way by the following definition: $T(u/\alpha) = (T(u))/\alpha$ for all $u \in \mathfrak{M}$, and all $\alpha \in \mathfrak{i}$, $\alpha \neq 0$. Of course, T induces an $\mathfrak{o}$-homomorphism of $\mathfrak{o}\mathfrak{M}$ into $\mathfrak{o}\mathfrak{N}$ so that we may think of the $\mathfrak{i}$-module $\mathfrak{L}$ of all $\mathfrak{i}$-homomorphisms of $\mathfrak{M}$ into $\mathfrak{N}$ as embedded in the $\mathfrak{o}$-module $\mathfrak{L}_{\mathfrak{o}}$ of all $\mathfrak{o}$-homomorphisms of $\mathfrak{o}\mathfrak{M}$ into $\mathfrak{o}\mathfrak{N}$, which in turn is embedded in the K-module $\mathfrak{L}_K$ of all K-homomorphisms of $K\mathfrak{M}$ into $K\mathfrak{N}$. One can easily verify that $\mathfrak{L}_K = K\mathfrak{L}$ and that $\mathfrak{L}_{\mathfrak{o}} = \mathfrak{o}\mathfrak{L}$. Now let us assume that $\mathfrak{M}$ and $\mathfrak{N}$ are finitely generated over $\mathfrak{i}$. For each $T \in \mathfrak{L}$, we define the determinant ideal $\delta(T)$ of T to be $(\mathfrak{N}: T(\mathfrak{M}))$.

If $\delta_{\mathfrak{o}}(T)$ and $\delta_K(T)$ denote the determinant ideals of T considered as an element of $\mathfrak{o}\mathfrak{L}$ and $K\mathfrak{L}$ respectively, then we have $\delta_{\mathfrak{o}}(T) = \mathfrak{o}\delta(T)$ and $\delta_K(T) = K\delta(T)$. Evidently, $T(\mathfrak{M}) = \mathfrak{N}$ if and only if $\delta(T) = \mathfrak{i}$, and if $\mathfrak{M}$ and $\mathfrak{N}$ have the same rank, T is biunique if and only if $\delta(T) \neq 0$, so that in this case, T is an isomorphism of $\mathfrak{M}$ onto $\mathfrak{N}$ if and only if $\delta(T) = \mathfrak{i}$. Furthermore, it is easily seen that in the case where $\mathfrak{M}$ and $\mathfrak{N}$ have the same rank, $\delta_{\mathfrak{o}}(T) = \mathfrak{o}.\,|T|$ and $\delta_K(T) = K.|T|$.

LEMMA 2. *If T_1 and T_2 are in $\mathfrak{L}$, if*

$$T_1 \equiv T_2 \pmod{\mathfrak{a}\mathfrak{L}}$$

where $\mathfrak{a}$ is an integral ideal of $\mathfrak{i}$, and if $\delta(T_1) + \mathfrak{a} = \mathfrak{i}$, then $\delta(T_2) + \mathfrak{a} = \mathfrak{i}$.

Proof. We may evidently exclude the case $\mathfrak{a} = 0$ where the statement is trivially true.

Since $\delta(T_1) + \mathfrak{a} = \mathfrak{i}$, $\mathfrak{N} = \mathfrak{i}\mathfrak{N} = \delta(T_1)\mathfrak{N} + \mathfrak{a}\mathfrak{N} \subseteq T_1(\mathfrak{M}) + \mathfrak{a}\mathfrak{N}$, so that $\mathfrak{N} = T_1(\mathfrak{M}) + \mathfrak{a}\mathfrak{N}$. But

$$T_1 \equiv T_2 \pmod{\mathfrak{a}\mathfrak{L}}$$

means that $T_1(\mathfrak{M}) + \mathfrak{a}\mathfrak{N} = T_2(\mathfrak{M}) + \mathfrak{a}\mathfrak{N}$, so that $\mathfrak{N} = T_2(\mathfrak{M}) + \mathfrak{a}\mathfrak{N}$. Let us assume that $\mathfrak{o}$ is the ring of all elements of K that are regular with respect to each of the prime ideal divisors of $\mathfrak{a}$. We have $\mathfrak{o}\mathfrak{N} = T_2(\mathfrak{o}\mathfrak{M}) + (\mathfrak{o}\mathfrak{a})(\mathfrak{o}\mathfrak{N})$.

Since $\mathfrak{o}$ is a principal ideal domain, there is a basis $u_1, \ldots, u_n$ of $\mathfrak{o}\mathfrak{N}$ over $\mathfrak{o}$ such that $\alpha_1 u_1, \ldots, \alpha_n u_n$ is a basis of $T_2(\mathfrak{o}\mathfrak{M})$ over $\mathfrak{o}$, where the α_i are in $\mathfrak{o}$. Then

$$\mathfrak{o}\mathfrak{N} = \sum_{i=1}^{n} (\mathfrak{o}\mathfrak{a} + (\alpha_i))u_i$$

so that $\mathfrak{o}\mathfrak{a} + (\alpha_i) = \mathfrak{o}$ for $i = 1, 2, \ldots, n$. But since

$$\frac{\mathfrak{o}\mathfrak{N}}{T_2(\mathfrak{o}\mathfrak{M})} \cong \sum_{i=1}^{n} \frac{\mathfrak{o}u_i}{\mathfrak{o}\alpha_i u_i} \qquad \text{(direct sum)},$$

$\delta_0(T_2) = \mathfrak{o}\alpha_1\alpha_2 \ldots \alpha_n$, so that $\mathfrak{o}(\delta(T_2) + \mathfrak{a}) = \delta_0(T_2) + \mathfrak{o}\mathfrak{a} = \mathfrak{o}$. Then, because of the choice of $\mathfrak{o}$, $\mathfrak{a} + \delta(T_2) = \mathfrak{i}$.

4. Representations of finite groups over Dedekind rings. Let A be an algebra with unity element, of finite dimension over K, let $\mathfrak{J}$ be an $\mathfrak{i}$-order of $\mathbf{A}$ and let $\mathfrak{O} = \mathfrak{o}\mathfrak{J}$.

We propose to study representations of $\mathfrak{J}$ by rings of $\mathfrak{i}$-endomorphisms of finitely generated torsion free $\mathfrak{i}$-modules, that map the unity element of $\mathfrak{J}$ onto the identity transformation, i.e., we propose to study $\mathfrak{J}$-modules that are finitely generated and torsion free over $\mathfrak{i}$.

If $\mathfrak{M}$ is such an $\mathfrak{J}$-module, then there is a unique way of defining $K\mathfrak{M}$ and $\mathfrak{o}\mathfrak{M}$ as an A-module and an $\mathfrak{O}$-module respectively so that they are extensions of the $\mathfrak{J}$-module $\mathfrak{M}$, explicitly, by setting $(a/\alpha)(u/\beta) = (au/\alpha\beta)$ for all $a \in \mathfrak{J}$, all $u \in \mathfrak{M}$ and all $\alpha, \beta \in \mathfrak{i}$, $\alpha \neq 0 \neq \beta$.

Two $\mathfrak{J}$-modules $\mathfrak{M}$ and $\mathfrak{N}$ will be said to be $\mathfrak{i}$-equivalent if they are $\mathfrak{J}$-isomorphic; they will be said to be K-equivalent if $K\mathfrak{M}$ and $K\mathfrak{N}$ are A-isomorphic and $\mathfrak{o}$-equivalent if $\mathfrak{o}\mathfrak{M}$ and $\mathfrak{o}\mathfrak{N}$ are $\mathfrak{O}$-isomorphic.

Let $\mathfrak{S}$ be a class of K-equivalent $\mathfrak{J}$-modules, complete in the sense that if an $\mathfrak{J}$-module $\mathfrak{M}$ is K-equivalent to an $\mathfrak{J}$-module in $\mathfrak{S}$, then $\mathfrak{M}$ is $\mathfrak{i}$-equivalent to an $\mathfrak{J}$-module in $\mathfrak{S}$, and let m denote the common rank of the $\mathfrak{i}$-modules in $\mathfrak{S}$. Let $\mathfrak{R}$ denote the partition of $\mathfrak{S}$ into complete subclasses of $\mathfrak{i}$-equivalent $\mathfrak{J}$-modules, and if $\mathfrak{o}$ is the ring of all elements of K that are regular with respect to each of the prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ of $\mathfrak{i}$, let $\mathfrak{R}(\mathfrak{p}_1, \ldots, \mathfrak{p}_r)$ denote the partition of $\mathfrak{S}$ into complete subclasses of $\mathfrak{o}$-equivalent $\mathfrak{J}$-modules. Finally, let $\mathbf{r}$ and $\mathbf{r}(\mathfrak{p}_1, \ldots, \mathfrak{p}_r)$ denote the number of subclasses of $\mathfrak{S}$ determined by $\mathfrak{R}$ and $\mathfrak{R}(\mathfrak{p}_1, \ldots, \mathfrak{p}_r)$ respectively.

THEOREM 5. *If $\mathfrak{M}$ and $\mathfrak{N}$ are two $\mathfrak{J}$-modules in $\mathfrak{S}$, then*

$$\mathfrak{M} \equiv \mathfrak{N} \pmod{\mathfrak{R}(\mathfrak{p}_1, \ldots, \mathfrak{p}_r)}$$

if and only if there is an $\mathfrak{J}$-homomorphism T of $\mathfrak{M}$ into $\mathfrak{N}$ for which $\delta(T) + \mathfrak{p}_i = \mathfrak{i}$ for $i = 1, 2, \ldots, r$.

Proof. Since $\mathfrak{M}$ and $\mathfrak{N}$ have the same rank,

$$\mathfrak{M} \equiv \mathfrak{N} \pmod{\mathfrak{R}(\mathfrak{p}_1, \ldots, \mathfrak{p}_r)}$$

if and only if there is an $\mathfrak{O}$-homomorphism T of $\mathfrak{o}\mathfrak{M}$ into $\mathfrak{o}\mathfrak{N}$ for which $(\mathfrak{o}\mathfrak{N}: T(\mathfrak{o}\mathfrak{M})) = \mathfrak{o}$, and we may evidently assume that T maps $\mathfrak{M}$ into $\mathfrak{N}$. But it is also evident that T is $\mathfrak{O}$-admissible if and only if it is $\mathfrak{J}$-admissible, when considered as a mapping of $\mathfrak{M}$ into $\mathfrak{N}$, and since $\delta_0(T) = (\mathfrak{o}\mathfrak{N}: T(\mathfrak{o}\mathfrak{M})) = \mathfrak{o}(\mathfrak{N}: T(\mathfrak{M})) = \mathfrak{o}\delta(T)$, $\delta_0(T) = \mathfrak{o}$ if and only if $\delta(T) + \mathfrak{p}_i = \mathfrak{i}$ for $i = 1, 2, \ldots, r$.

THEOREM 6. *If each prime ideal $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ is different from each prime ideal $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$, then*

$$\mathfrak{R}(\mathfrak{p}_1, \ldots, \mathfrak{p}_r, \mathfrak{q}_1, \ldots, \mathfrak{q}_s) = \mathfrak{R}(\mathfrak{p}_1, \ldots, \mathfrak{p}_r) \cap \mathfrak{R}(\mathfrak{q}_1, \ldots, \mathfrak{q}_s)$$

Proof. It is clear, from the preceding theorem, that

$$\mathfrak{R}(\mathfrak{p}_1, \ldots, \mathfrak{p}_r, \mathfrak{q}_1, \ldots, \mathfrak{q}_s) \subseteq \mathfrak{R}(\mathfrak{p}_1, \ldots, \mathfrak{p}_r) \cap \mathfrak{R}(\mathfrak{q}_1, \ldots, \mathfrak{q}_s)$$

Suppose that $\mathfrak{M}$ and $\mathfrak{N}$ are $\mathfrak{J}$-modules in $\mathfrak{S}$ and that

$$\mathfrak{M} \equiv \mathfrak{N} \pmod{\mathfrak{R}(\mathfrak{p}_1, \ldots, \mathfrak{p}_r) \cap \mathfrak{R}(\mathfrak{q}_1, \ldots, \mathfrak{q}_s)}$$

This means that there are $\mathfrak{J}$-homomorphisms T_1 and T_2 of $\mathfrak{M}$ into $\mathfrak{N}$ for which

$$\delta(T_1) + \mathfrak{p}_i = \mathfrak{i} (i = 1, 2, \ldots, r), \quad \delta(T_2) + \mathfrak{q}_j = \mathfrak{i} \ (j = 1, 2, \ldots, s).$$

Since $\mathfrak{p}_1\mathfrak{p}_2 \ldots \mathfrak{p}_r$ is relatively prime to $\mathfrak{q}_1\mathfrak{q}_2 \ldots \mathfrak{q}_s$, we may set $1 = \alpha_1 + \alpha_2$, where $\alpha_1 \in \mathfrak{p}_1\mathfrak{p}_2 \ldots \mathfrak{p}_r$ and $\alpha_2 \in \mathfrak{q}_1\mathfrak{q}_2 \ldots \mathfrak{q}_s$. Then if we set $T = \alpha_1 T_2 + \alpha_2 T_1$, T is an $\mathfrak{J}$-homomorphism of $\mathfrak{M}$ into $\mathfrak{N}$, and furthermore, if $\mathfrak{L}$ denotes the $\mathfrak{i}$-module of all $\mathfrak{J}$-homomorphisms of $\mathfrak{M}$ into $\mathfrak{N}$,

$$T \equiv T_1 \pmod{\mathfrak{p}_1\mathfrak{p}_2 \ldots \mathfrak{p}_r\mathfrak{L}}, \qquad T \equiv T_2 \pmod{\mathfrak{q}_1\mathfrak{q}_2 \ldots \mathfrak{q}_s\mathfrak{L}},$$

so that by Lemma 2,

$$\delta(T) + \mathfrak{p}_i = \mathfrak{i} \ (i = 1, 2, \ldots, r), \qquad \delta(T) + \mathfrak{q}_j = \mathfrak{i} \ (j = 1, 2, \ldots, s).$$

Therefore

$$\mathfrak{M} \equiv \mathfrak{N} \pmod{\mathfrak{R}(\mathfrak{p}_1, \ldots, \mathfrak{p}_r, \mathfrak{q}_1, \ldots, \mathfrak{q}_s)}$$

THEOREM 7. *If $\mathfrak{M}$ and $\mathfrak{N}$ are $\mathfrak{J}$-modules in $\mathfrak{S}$, and if each prime ideal $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ is different from each prime ideal $\mathfrak{q}_1, \ldots, \mathfrak{q}_s$, then there is an $\mathfrak{J}$-module $\mathfrak{M}' \in \mathfrak{S}$ such that*

$$\mathfrak{M} \equiv \mathfrak{M}' \pmod{\mathfrak{R}(\mathfrak{p}_1, \ldots, \mathfrak{p}_r)}, \quad \mathfrak{M}' \equiv \mathfrak{N} \pmod{\mathfrak{R}(\mathfrak{q}_1, \ldots, \mathfrak{q}_s)}.$$

Proof. Since $\mathfrak{M}$ and $\mathfrak{N}$ are K-equivalent, there is an A-isomorphism T of $K\mathfrak{M}$ onto $K\mathfrak{N}$, and we may evidently assume that T maps $\mathfrak{M}$ into $\mathfrak{N}$.

Let $\mathfrak{o}$ be the ring of all elements of K that are regular with respect to the prime ideal divisors of $\delta(T) = (\mathfrak{N} : T(\mathfrak{M}))$.

Since $\mathfrak{o}$ is a principal ideal domain, there is a basis $u_1, \ldots, u_m$ of $\mathfrak{o}\mathfrak{N}$ such that $\alpha_1 u_1, \ldots, \alpha_m u_m$ is a basis of $T(\mathfrak{o}\mathfrak{M})$, where the α_i are in $\mathfrak{o}$, and furthermore, since T is non-singular, $\delta_{\mathfrak{O}}(T) = \mathfrak{o}\alpha_1\alpha_2 \ldots \alpha_m \neq 0$. We may even assume that the α_i are in $\mathfrak{i}$.

Let $\mathfrak{o}\alpha_i = \mathfrak{B}_i\mathfrak{C}_i$, where the $\mathfrak{B}_i$ and $\mathfrak{C}_i$ are integral ideals of $\mathfrak{o}$ such that $\mathfrak{B}_i + \mathfrak{C}_i = \mathfrak{o}$, $\mathfrak{B}_i + \mathfrak{o}\mathfrak{p}_j = \mathfrak{o}$ and $\mathfrak{C}_i + \mathfrak{o}\mathfrak{q}_k = \mathfrak{o}$ for $i = 1, 2, \ldots, m$, $j = 1, 2, \ldots, r$ and $k = 1, 2, \ldots, s$, Let $\mathfrak{B}$ denote the intersection of the $\mathfrak{B}_i$. Then since

$$\mathfrak{O}(\mathfrak{B}^{-1}T(\mathfrak{o}\mathfrak{M})) = \mathfrak{B}^{-1}(\mathfrak{O}T(\mathfrak{o}\mathfrak{M})) \subseteq \mathfrak{B}^{-1}T(\mathfrak{o}\mathfrak{M}),$$

$\mathfrak{B}^{-1}T(\mathfrak{o}\mathfrak{M})$ is an $\mathfrak{O}$-module.

Let $\mathfrak{M}' = \mathfrak{N} \cap \mathfrak{B}^{-1}T(\mathfrak{o}\mathfrak{M})$. Then $\mathfrak{M}'$ is surely an $\mathfrak{J}$-module, and since

$$\mathfrak{o}\mathfrak{M}' = \mathfrak{o}\mathfrak{N} \cap \mathfrak{B}^{-1}T(\mathfrak{o}\mathfrak{M}) = \sum_{i=1}^{m} (\mathfrak{o} \cap \mathfrak{B}^{-1}\alpha_i)u_i$$

$\mathfrak{o}\mathfrak{M}'$ has the same rank as $\mathfrak{o}\mathfrak{N}$ so that $\mathfrak{M}'$ has the same rank as $\mathfrak{N}$ and therefore, $\mathfrak{M}'$ is an $\mathfrak{J}$-module in $\mathfrak{S}$. But since $\mathfrak{B}\mathfrak{B}_i^{-1} + \mathfrak{C}_i = \mathfrak{o}$, $\mathfrak{B}\alpha_i^{-1} + \mathfrak{o} = \mathfrak{C}_i^{-1}$, so that $\mathfrak{B}^{-1}\alpha_i \cap \mathfrak{o} = \mathfrak{C}_i$ and therefore

$$\mathfrak{o}\mathfrak{M}' = \sum_{i=1}^{m} \mathfrak{C}_i u_i$$

Since $T(\mathfrak{M}) \subseteq \mathfrak{M}'$, T may be considered as an $\mathfrak{J}$-homomorphism of $\mathfrak{M}$ into $\mathfrak{M}'$ whose determinant ideal is $(\mathfrak{M}' : T(\mathfrak{M}))$. But $\mathfrak{o}(\mathfrak{M}' : T(\mathfrak{M})) = (\mathfrak{o}\mathfrak{M}' : T(\mathfrak{o}\mathfrak{M})) = \mathfrak{B}_1\mathfrak{B}_2 \dots \mathfrak{B}_m$ so that

$$\mathfrak{o}((\mathfrak{M}' : T(\mathfrak{M})) + \mathfrak{p}_j) = \mathfrak{o}(\mathfrak{M}' : T(\mathfrak{M})) + \mathfrak{o}\mathfrak{p}_j = \mathfrak{B}_1\mathfrak{B}_2 \dots \mathfrak{B}_m + \mathfrak{o}\mathfrak{p}_j = \mathfrak{o},$$

and therefore, because of the choice of $\mathfrak{o}$,

$$(\mathfrak{M}' : T(\mathfrak{M})) + \mathfrak{p}_j = \mathfrak{i}$$

and this for $j = 1, 2, \dots, r$. Therefore

$$\mathfrak{M} \equiv \mathfrak{M}' \pmod{\mathfrak{N}(\mathfrak{p}_1, \dots, \mathfrak{p}_r)}$$

Furthermore, the determinant ideal of the identity mapping of $\mathfrak{M}'$ into $\mathfrak{N}$ is $(\mathfrak{N} : \mathfrak{M}')$ and since

$$\begin{aligned} \mathfrak{o}(\mathfrak{N} : \mathfrak{M}') = (\mathfrak{o}\mathfrak{N} : \mathfrak{o}\mathfrak{M}') &= \mathfrak{C}_1\mathfrak{C}_2 \dots \mathfrak{C}_m, \ \mathfrak{o}((\mathfrak{N} : \mathfrak{M}') + \mathfrak{q}_k) \\ &= \mathfrak{C}_1\mathfrak{C}_2 \dots \mathfrak{C}_m + \mathfrak{o}\mathfrak{q}_k = \mathfrak{o} \end{aligned}$$

so that by the choice of $\mathfrak{o}$,

$$((\mathfrak{N} : \mathfrak{M}') + \mathfrak{q}_k) = \mathfrak{i}, \qquad k = 1, 2, \dots, s,$$

and therefore

$$\mathfrak{M}' \equiv \mathfrak{N} \pmod{\mathfrak{N}(\mathfrak{q}_1, \dots, \mathfrak{q}_s)}.$$

COROLLARY.

$$\mathfrak{r}(\mathfrak{p}_1, \dots, \mathfrak{p}_r, \mathfrak{q}_1, \dots, \mathfrak{q}_s) = \mathfrak{r}(\mathfrak{p}_1, \dots, \mathfrak{p}_r) \cdot \mathfrak{r}(\mathfrak{q}_1, \dots, \mathfrak{q}_s)$$

Theorems 6 and 7 may evidently be extended by induction so that

$$\mathfrak{N}(\mathfrak{p}_1, \dots, \mathfrak{p}_r) = \bigcap_{i=1}^{r} \mathfrak{N}(\mathfrak{p}_i)$$

and

$$\mathfrak{r}(\mathfrak{p}_1, \dots, \mathfrak{p}_r) = \prod_{i=1}^{r} \mathfrak{r}(\mathfrak{p}_i)$$

DEFINITION. *Two $\mathfrak{J}$-modules $\mathfrak{M}$ and $\mathfrak{N}$ will be said to be of the same genus*[2] *if $\mathfrak{M}$ and $\mathfrak{N}$ are $\mathfrak{o}$-equivalent for every ring $\mathfrak{o}$ of all elements of K that are regular with respect to a single prime ideal of $\mathfrak{i}$.*

[2] We could say that $\mathfrak{M}$ and $\mathfrak{N}$ are of the same genus if the corresponding representations are unimodularly equivalent over all $\mathfrak{P}$-adic extensions of K, but because of Corollary 1 of Theorem 1, this is equivalent to the above definition.

Let $\mathfrak{R}_g$ denote the partition of $\mathfrak{S}$ into complete subclasses of $\mathfrak{J}$-modules of the same genus. Then, by definition,

$$\mathfrak{R}_g = \bigcap_{\mathfrak{p}} R(\mathfrak{p})$$

where the intersection extends over all prime ideals $\mathfrak{p}$ of $\mathfrak{i}$. Let $\mathfrak{r}_g$ denote the number of subclasses of $\mathfrak{S}$ determined by $\mathfrak{R}_g$.

From now on, we will assume that A, $\mathfrak{O}$ and $\mathfrak{J}$ are the group rings of the finite group $\mathfrak{G}$ of order N over K, $\mathfrak{o}$ and $\mathfrak{i}$ respectively.

Let $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ be the prime ideal divisors of N in $\mathfrak{i}$. By Theorem 4, we know that if $\mathfrak{p}$ does not divide N and if $\mathfrak{o}$ is the ring of all $\mathfrak{p}$-regular elements of K, then two $\mathfrak{J}$-modules are $\mathfrak{o}$-equivalent if and only if they are K-equivalent, so that in this case, $\mathfrak{R}(\mathfrak{p})$ determines only one subclass of $\mathfrak{S}$, namely $\mathfrak{S}$ itself. Therefore

$$\mathfrak{R}_g = \bigcap_{i=1}^{r} \mathfrak{R}(\mathfrak{p}_i) = \mathfrak{R}(\mathfrak{p}_1, \ldots, \mathfrak{p}_r), \qquad \mathfrak{r}_g = \prod_{i=1}^{r} \mathfrak{r}(\mathfrak{p}_i).$$

Now let $\mathfrak{a}$ be a non-null integral ideal of $\mathfrak{i}$ and let $\mathfrak{M}$ be an $\mathfrak{J}$-module in $\mathfrak{S}$. Since $\mathfrak{J}(\mathfrak{a}\mathfrak{M}) = \mathfrak{a}(\mathfrak{J}\mathfrak{M}) \subseteq \mathfrak{a}\mathfrak{M}$, $\mathfrak{a}\mathfrak{M}$ is an $\mathfrak{J}$-submodule of $K\mathfrak{M}$, and since $\mathfrak{a}\mathfrak{M}$ has the same rank as $\mathfrak{M}$, $\mathfrak{a}\mathfrak{M} \in \mathfrak{S}$. Then there is a non-null integral ideal $\mathfrak{b}$ of $\mathfrak{i}$ such that $\mathfrak{a}\mathfrak{b} = (\alpha), \alpha \in \mathfrak{i}, \alpha \neq 0$ and $\mathfrak{b} + (N) = \mathfrak{i}$. The function T mapping each $u \in \mathfrak{M}$ onto αu is an $\mathfrak{J}$-homomorphism of $\mathfrak{M}$ into $\mathfrak{a}\mathfrak{M}$ and since

$$(\mathfrak{M} : \alpha\mathfrak{M}) = (\mathfrak{M} : \mathfrak{a}\mathfrak{M})(\mathfrak{a}\mathfrak{M} : \alpha\mathfrak{M}), \qquad \alpha^m = \mathfrak{a}^m \cdot \delta(T),$$

so that $\delta(T) = \mathfrak{b}^m$ and therefore

$$\mathfrak{M} \equiv \mathfrak{a}\mathfrak{M} \pmod{\mathfrak{R}_g}.$$

Now let us assume that the $\mathfrak{J}$-modules $\mathfrak{M} \in \mathfrak{S}$ are such that the representations of A belonging to $K\mathfrak{M}$ are absolutely irreducible. Let $\mathfrak{M}$ and $\mathfrak{N}$ be two $\mathfrak{J}$-modules in $\mathfrak{S}$ and suppose that

$$\mathfrak{M} \equiv \mathfrak{N} \pmod{\mathfrak{R}_g}.$$

This means that there is an $\mathfrak{J}$-homomorphism T of $\mathfrak{M}$ into $\mathfrak{N}$ for which $\delta(T) + (N) = \mathfrak{i}$. Let

$$\delta(T) = \mathfrak{q}_1^{f_1}\mathfrak{q}_2^{f_2} \ldots \mathfrak{q}_s^{f_s}$$

be the decomposition of $\delta(T)$ into prime ideals in $\mathfrak{i}$. Since the representations of $\mathfrak{G}$ belonging to $K\mathfrak{M}$ and $K\mathfrak{N}$ are absolutely irreducible, they are irreducible over the $\mathfrak{P}$-adic extensions of K determined by the $\mathfrak{q}_j$, so that by the corollary to Theorem 3, each f_j is divisible by m and

$$T \in \mathfrak{q}_j^{f_j/m} \cdot \mathfrak{L}.$$

If we set

$$\mathfrak{b} = \mathfrak{q}_1^{f_1/m} \cdot \mathfrak{q}_2^{f_2/m} \ldots \mathfrak{q}_s^{f_s/m},$$

we have $T \in \mathfrak{b}\mathfrak{L}$ and $\delta(T) = \mathfrak{b}^m$. Therefore $T(\mathfrak{M}) \subseteq \mathfrak{b}\mathfrak{N}$ and since

$$\mathfrak{b}^m = \delta(T) = (\mathfrak{N}: T(\mathfrak{M}) = (\mathfrak{N}: \mathfrak{b}\mathfrak{N})(\mathfrak{b}\mathfrak{N}: T(\mathfrak{M})) = \mathfrak{b}^m(\mathfrak{b}\mathfrak{N}: T(\mathfrak{M})),$$
$$(\mathfrak{b}\mathfrak{N}: T(\mathfrak{M})) = \mathfrak{i},$$

so that $T(\mathfrak{M}) = \mathfrak{b}\mathfrak{N}$.

Let $\mathfrak{S}'$ be a complete subclass of $\mathfrak{S}$ determined by $\mathfrak{N}_\varrho$ and let $\mathfrak{M}$ be an $\mathfrak{J}$-module in $\mathfrak{S}'$. By the preceding argument, we may assume that $\mathfrak{S}'$ just consists of all the $\mathfrak{J}$-modules $\mathfrak{b}\mathfrak{M}$ where $\mathfrak{b}$ is a non-null integral ideal of $\mathfrak{i}$. If $\mathfrak{b}$ and $\mathfrak{b}'$ are two such ideals of $\mathfrak{i}$ and if $\mathfrak{b}$ and $\mathfrak{b}'$ are in the same ideal class, then there is an $\alpha \in K$, $\alpha \neq 0$ such that the mapping $\beta \to \alpha\beta$, for all $\beta \in \mathfrak{b}$, is an $\mathfrak{i}$-isomorphism of $\mathfrak{b}$ onto $\mathfrak{b}'$. Then it is easily seen that the mapping $u \to \alpha u$, for all $u \in \mathfrak{b}\mathfrak{M}$, is an $\mathfrak{J}$-isomorphism of $\mathfrak{b}\mathfrak{M}$ onto $\mathfrak{b}'\mathfrak{M}$.

Conversely, if T is an $\mathfrak{J}$-isomorphism of $\mathfrak{b}\mathfrak{M}$ onto $\mathfrak{b}'\mathfrak{M}$, since the representation of A belonging to $K\mathfrak{M} = K\mathfrak{b}\mathfrak{M} = K\mathfrak{b}'\mathfrak{M}$ is absolutely irreducible, it follows that T multiplies each element of $\mathfrak{b}\mathfrak{M}$ by some scalar $\alpha \in K$, $\alpha \neq 0$, so that $\mathfrak{b}'\mathfrak{M} = \alpha\mathfrak{b}\mathfrak{M}$ and therefore, $\mathfrak{b}' = \alpha\mathfrak{b}$, i.e., $\mathfrak{b}$ and $\mathfrak{b}'$ are in the same ideal class.

THEOREM 8. *If each $\mathfrak{J}$-module $\mathfrak{M} \in \mathfrak{S}$ is such that the representation of A belonging to $K\mathfrak{M}$ is absolutely irreducible, then $\mathfrak{r} = h\mathfrak{r}_\varrho$, where h is the number of classes of isomorphic ideals of $\mathfrak{i}$.*

If $\mathfrak{i}$ is the ring of algebraic integers of an algebraic number field K, then h is finite and also, since each integral ideal of $\mathfrak{i}$ has a finite index in $\mathfrak{i}$, by Lemma 2 of Theorem 1, each $\mathfrak{r}(\mathfrak{p}_i)$ is finite, so that if $\mathfrak{S}$ satisfies the condition of Theorem 8, the number

$$\mathfrak{r} = \mathfrak{h}\prod_{i=1}^{r} \mathfrak{r}(\mathfrak{p}_i)$$

is finite.

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NOTE ON A MATRIX THEOREM OF A. BRAUER AND ITS EXTENSION

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1. Introduction. In one of his papers on limits for the characteristic roots of a Matrix, Brauer (1) has stated a theorem, which connects the roots of a given square matrix A, with those of a matrix A^* derived from A by a certain process. The proof of this theorem involves a continuity argument and in a recent paper on the construction of stochastic matrices Hazel Perfect (5) has given a proof which avoids considerations of continuity. However, her proof, involving several multiple derivatives (not with respect to the elements of the matrix), is unnecessarily heavy, and in the present note I give a proof which is simple, short and avoids both continuity and differentiation.

Two extensions of Brauer's theorem are then considered. In each matrix A^* is of the form $A^* = A + XK'$ where X is an $n \times r$ matrix (n being the order of A) whose columns are latent vectors of A, and K is an arbitrary $n \times r$ matrix. These extensions arise according as the columns of X are associated with the same latent root of A, or different roots.

2. Brauer's theorem. In what follows, symbols in bold type represent column vectors. A row vector is represented by an attached prime, which is also used to denote the transpose of a matrix. The unit matrix is denoted by I. Brauer's result may be stated as follows:

THEOREM 1. *Let the square matrix A of order n have latent roots $\lambda_1, \ldots, \lambda_s$ with multiplicities $m_1, \ldots, m_s$; let $\mathbf{x}$ be a latent column vector of A associated with the root λ_1, and let $\mathbf{k}$ be an arbitrary column vector. Then the matrix $A^* = A + \mathbf{x}\mathbf{k}'$ has latent roots $\lambda_1 + \mathbf{k}'\mathbf{x}, \lambda_1, \ldots, \lambda_s$ with multiplicities $1, m_1 - 1, m_2, \ldots, m_s$.*

Proof. We have

$$|\lambda I - A| = \prod_{i=1}^{s} (\lambda - \lambda_i)^{m_i}$$

and also $A\mathbf{x} = \lambda_1\mathbf{x}$. Now, since $\mathbf{x}\mathbf{k}'$ is of rank 1, we have

$$|\lambda I - A^*| = |\lambda I - A - \mathbf{x}\mathbf{k}'|$$

$$= |\lambda I - A| - \sum_{i=1}^{n} k_i |\mathbf{a}_1, \ldots, \mathbf{a}_{i-1}, \mathbf{x}, \mathbf{a}_{i+1}, \ldots, \mathbf{a}_n|,$$

where, in the last determinant, we have written $\mathbf{a}_1, \ldots, \mathbf{a}_n$ for the columns of $\lambda I - A$.

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If we write $A_{ij}(\lambda)$ for the co-factor of the (i, j)th element of $\lambda I - A$, we have

$$|\lambda I - A^*| = |\lambda I - A| - \sum_{i=1}^{n} k_i \sum_{j=1}^{n} x_j A_{ji}(\lambda).$$

Now $A_{ji}(\lambda)$ is the (i, j)th element of $\operatorname{adj}(\lambda I - A)$. Also, from the well-known identity

$$\operatorname{adj}(\lambda I - A) \,.\, (\lambda I - A) = |\lambda I - A| I,$$

it follows at once that

$$(\lambda - \lambda_1) \operatorname{adj}(\lambda I - A) \,.\, \mathbf{x} = |\lambda I - A| \mathbf{x}.$$

This gives

$$\sum_{j=1}^{n} A_{ji}(\lambda) x_j = x_i F(\lambda) \qquad (i = 1, \ldots, n),$$

where $F(\lambda) = |\lambda I - A|/(\lambda - \lambda_1)$. Hence

$$\begin{aligned} |\lambda I - A^*| &= |\lambda I - A| - (\mathbf{k}'\mathbf{x})\, F(\lambda) \\ &= \{\lambda - \lambda_1 - (\mathbf{k}'\mathbf{x})\}(\lambda - \lambda_1)^{m_1 - 1}(\lambda - \lambda_2)^{m_2} \ldots (\lambda - \lambda_s)^{m_s} \end{aligned}$$

and this proves the theorem.

3. Extensions. The effect of Brauer's modification of the matrix A is to bring about a "splitting" of the spectrum of latent roots, the new root $\lambda_1 + \mathbf{k}'\mathbf{x}$ differing from λ_1 by an infinitesimal provided the elements of $\mathbf{k}$ are small quantities of the first order. It is a natural question to ask how far this splitting process may be carried, and consideration of this question leads to an extension of Brauer's theorem.

We begin with the well-known

DEFINITION. *Two square matrices A and B, of the same order n, are said to possess property P (the Frobenius property) if the characteristic roots of A and B, say $\alpha_1, \ldots, \alpha_n$ and $\beta_1, \ldots, \beta_n$ may be so ordered that the characteristic roots of any polynomial $f(A, B)$ are $f(\alpha_i, \beta_i)$ $(i = 1, \ldots, n)$.*

It is known (2; 4) that a necessary and sufficient condition that A and B possess property P is that the matrix $AB - BA$ belong to the radical of the algebra generated over the base field by A and B. We shall prove

THEOREM 2. *Let A be a square matrix of order n and let $B = XK'$ where* (i) *X is an $n \times r$ matrix whose column vectors are characteristic vectors (independent or otherwise) of A associated with the same latent root λ,* (ii) *K is an arbitrary $n \times r$ matrix. Then A and B have property P.*

Proof. It is sufficient to prove that $f(A, B)(AB - BA)$ is nilpotent for all polynomials $f(A, B)$. Write $N = AB - BA$. Since the column vectors of X are all characteristic vectors of A associated with the root λ, we have

$$AX = \lambda X.$$

Thus $AB = (AX)K' = \lambda B$, $N = AB - BA = B(\lambda I - A)$. It follows that $N^2 = 0$.

Next, $NB = 0$ and, for the polynomial $f(A, B)$, we have

$$f(A, B)B = f(\lambda I, B)B, \quad f(A, B)N = f(\lambda I, B)N.$$

Thus $[f(A, B)N]^2 = [f(\lambda I, B)N]^2 = 0$.

We now use Theorem 2 to obtain an extension of Brauer's theorem. Let X have rank $s (s \leqslant r)$. Then B has at most s non-zero characteristic roots. Since A and B have property P and $AB = \lambda B$ the non-zero roots of B all associate with the root λ of A. Now suppose the roots of B are $\mu_1, \ldots, \mu_\sigma$ $(\sigma \leqslant s, \mu_i \neq 0)$ and a zero root of multiplicity $n - \sigma$; and let the roots of A be $\lambda, \lambda_1, \ldots, \lambda_t$ of multiplicities $\tau, m_1, \ldots, m_t$ respectively. Since the μ_i all associate with λ we have $\sigma \leqslant \tau$. Theorem 2 now leads to

THEOREM 3. *If A and B are the matrices occurring in Theorem 2 and $f(A, B)$ is any polynomial in A and B, then $f(\lambda, \mu_i)(i = 1, \ldots, \sigma)$ are roots of $f(A, B)$ and the remaining roots are $f(\lambda, 0)$, $f(\lambda_1, 0), \ldots, f(\lambda_t, 0)$ with multiplicities $\tau - \sigma, m_1, \ldots, m_t$ respectively.*

On putting $f(A, B) = A + B$ and noting that the non-zero roots of XK' and $K'X$ are the same we obtain

THEOREM 4. *If $A^* = A + XK'$, where X and K are as defined in Theorem 2, and if the non-zero roots of $K'X$ are $\mu_1, \ldots, \mu_\sigma$ $(\sigma \leqslant s \leqslant r)$ then the roots of A^* are $\mu_i + \lambda (i = 1, \ldots, \sigma)$ and $\lambda, \lambda_1, \ldots, \lambda_t$ with multiplicities $\tau - \sigma$, $m_1, \ldots, m_t$ respectively.*

Another extension of Brauer's theorem is obtained by taking the columns of X to be latent vectors of A associated with distinct latent roots $\lambda_1, \ldots, \lambda_r$ of A. By a proper choice of basis, that is, by a suitable similarity transformation, A and X may be simultaneously reduced to the forms

$$A = \begin{pmatrix} \Lambda & A_1 \\ 0 & A_2 \end{pmatrix}, \quad X = \begin{pmatrix} I_r \\ 0 \end{pmatrix}$$

where $\Lambda = \operatorname{diag}(\lambda_1, \ldots, \lambda_r)$ and I_r is the unit matrix of order r.

If $K = (K_1, K_2)$ where K_1 is of order r, we have $K'X = K_1$ and

$$A + XK' = \begin{pmatrix} \Lambda + K_1 & A_1 + K_2 \\ 0 & A_2 \end{pmatrix}.$$

From this there follows:

THEOREM 5. *If $A^* = A + XK'$, where the column vectors of the $n \times r$ matrix X are latent vectors of A associated with distinct latent roots $\lambda_1, \ldots, \lambda_r$ of A, then the numbers $\epsilon_1, \ldots, \epsilon_r$ are latent roots of A^*, where $\epsilon_1, \ldots, \epsilon_r$ are the roots of $\Lambda + K'X$; also, every root of A^* other then $\epsilon_1, \ldots, \epsilon_r$ is a root of A, with the same multiplicity.*

4. Matrices having the same characteristic equation. In Theorem 5 let K be such that

$$K'X = PTP^{-1},$$

where P is a permutation matrix and T is a matrix whose elements t_{ij} satisfy $t_{ij} = 0$ for $i \leqslant j$ or $t_{ij} = 0$ for $i \geqslant j$, that is, T is lower or upper nilpotent triangular. Then

$$\Lambda + K'X = P(P^{-1}\Lambda P + T)P^{-1}.$$

Now $P^{-1}\Lambda P$ is a diagonal matrix and the diagonal of T contains only zeros. Thus Λ and $\Lambda + K'X$ have the same characteristic equation, and hence, by Theorem 5, A and $A + XK'$ have this property also.

It is perhaps worth pointing out that this result follows from the following theorem proved recently (3).

THEOREM. *Let A and B be matrices of orders n and r, such that there exists an $n \times r$ matrix X, of rank r, for which $AX = XB$. If K is any $n \times r$ matrix, then the pair of matrices A, XK' has property P if and only if the pair B, $K'X$ has this property.*

In the case at hand we have $AX = X\Lambda$, that is $B = \Lambda$. Now if $K'X = PTP^{-1}$ then Λ and $K'X$ have property P, since $P^{-1}\Lambda P$ and T have this property. Hence by this theorem A and XK' have this property. Since XK' is nilpotent it follows that A and $A + XK'$ have the same characteristic equation.

In conclusion, I wish to thank the referee for several suggestions leading to improved proofs of the various theorems.

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SOME RESULTS ON THE ASYMPTOTIC BEHAVIOR OF LINEAR SYSTEMS

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1. Introduction. We consider first in §2 the asymptotic behavior as $t \to \infty$ of the solutions of the vector-matrix differential equation

$$\dot{x} = \{A + B(t)\}x, \tag{1.1}$$

where A is a constant n-square complex matrix, $B(t)$ a continuous complex valued n-square matrix defined on $[0, \infty)$, and x a complex n-vector.

It is readily shown **(4)** that the asymptotic behavior of solutions to (1.1) can be made to depend on the functions $\lambda_M\{A + A^* + B(t) + B^*(t)\}$ and $\lambda_m\{A + A^* + B(t) + B^*(t)\}$ where $A^* = \bar{A}'$ and λ_M, λ_m are respectively the maximum and minimum eigenvalues of the indicated Hermitian matrix. We recapitulate this brief calculation in §2.

There are two types of theorems concerning (1.1) in the sequel: (i) A arbitrary with hypotheses on the eigenvalues of $(A + A^*)$; (ii) A triangular with hypotheses on the real parts of the eigenvalues of A. In both (i) and (ii) less than the absolute integrability of the functions $B_{ij}(t)$ is required **(1, pp. 32–63)**.

In §3 we discuss the behavior as $t \to \infty$ of solutions to the equation

$$\dot{x} = \{A(t) + B(t)\}x, \tag{1.2}$$

in which the entries of $A(t)$ are continuous complex-valued almost-periodic functions. The main result concerning (1.2) depends on a theorem of Favard which will be stated. In this case, however, it becomes necessary to assume the absolute integrability of all entries of $B(t)$.

We set

$$||x||^2 = \sum_{i=1}^{n} |x_i|^2;$$

boundedness refers to this norm. Also let $\Re(X) = (X + \bar{X})/2$ and $\Im(X) = (X - \bar{X})/2i$. $||X||^2$ = trace (XX^*), α' = transpose of α. We note the following two elementary results that are subsequently used:

I. *If X and Y are Hermitian n-square matrices then*

$$\lambda_M(X + Y) \leqslant \lambda_M(X) + \lambda_M(Y), \tag{1.3}$$

$$\lambda_m(X + Y) \geqslant \lambda_m(X) + \lambda_m(Y). \tag{1.4}$$

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This follows immediately upon noting that $X + Y$ is Hermitian and

$$\lambda_M(X + Y) = \max_{t=1} z^*(X + Y)z \leqslant \max_{t=1} z^*Xz + \max_{t=1} z^*Yz$$
$$= \lambda_M(X) + \lambda_M(Y),$$

where $t = ||z||^2$. Similarly for (1.4).

The following well-known device is due to O. Perron.

II. *If X has eigenvalues $\lambda_1, \ldots, \lambda_n$ then for any $\epsilon > 0$ there exists a matrix $D(\epsilon)$ similar to X such that $D_{ii}(\epsilon) = \lambda_i$ and $|D_{ij}(\epsilon)| < \epsilon$ for $i \neq j$.*

For assume X is in Jordan form, set

$$H = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & \epsilon & & 0 \\ \cdot & & \cdot & \\ \cdot & & \cdot & \\ \cdot & & & 0 \\ 0 & \cdots & 0 & \epsilon^{n-1} \end{pmatrix}$$

and note that

$$H^{-1}XH = \begin{pmatrix} \lambda_1 & \epsilon & 0 & \cdots & 0 \\ 0 & \cdot & & & \cdot \\ \cdot & & \cdot & & \cdot \\ \cdot & & & \cdot & \\ \cdot & & & & 0 \\ \cdot & & & & \epsilon \\ 0 & \cdot & & 0 & \lambda_n \end{pmatrix}$$

2. The equation (1.1). In discussing (1.1) we, of course, omit the trivial solution $x(t) = 0$. We assume that the starting time of every solution is $t_0 = 0$ since any solution $x(t)$ with starting time $t_0 > 0$ may be continued over $[0, t_0]$.

THEOREM 1. *Consider* (1.1) *with A arbitrary. Assume*

$$\lambda_M(A + A^*) = \omega \tag{2.1}$$

and there exists L such that $t > L$ implies either (a):

$$\frac{1}{t}\int_0^t \max_i \Re\{B_{ii}(s)\}ds < -\tfrac{1}{2}\omega \tag{2.2}$$

$$\int_0^\infty |\Re\{B_{ij}(s)\}|ds < \infty, \quad \int_0^\infty |\Im\{B(s) - B'(s)\}_{ij}|ds < \infty \tag{2.3}$$

for $i \neq j$, or (b):

$$\frac{1}{t}\int_0^t \left(\max_i \Re\{B_{ii}(s)\} + \sum_{i \neq j} |\Re\{B_{ij}(s)\}| + |\Im\{B(s) - B'(s)\}_{ij}|\right)ds$$
$$< -\tfrac{1}{2}\omega; \tag{2.4}$$

then in both cases every solution of (1.1) *is uniformly bounded as* $t \to \infty$. *If in either* (2.2) *or* (2.4) *the left sides are bounded strictly below* $-\frac{1}{2}\omega$, *then every solution converges to* 0 *as* $t \to \infty$.

Proof. Taking the inner product on the left with x^* in (1.1) we obtain

$$x^*\dot{x} = x^*\{A + B(t)\}x \tag{2.5}$$

and

$$\frac{d}{dt}||x||^2 = x^*\dot{x} + \dot{x}^*x = x^*\{A + A^* + B(t) + B^*(t)\}x. \tag{2.6}$$

The matrix on the right in (2.6) is Hermitian for all t and hence let $U(t)$ be a unitary matrix reducing it to canonical form. The substitution $x = U(t)z$ then yields

$$\frac{d}{dt}||z||^2 = z^*\text{diagonal } \lambda_i\{A + A^* + B(t) + B^*(t)\}z$$

$$= \sum_{i=1}^{n} \lambda_i|z_i|^2 = ||z||^2\sum_{i=1}^{n} \lambda_i\delta_i$$

where $\delta_i = |z_i|^2/||z||^2$, $\sum\delta_i = 1$, $0 \leqslant \delta_i \leqslant 1$. Integrating we obtain

$$||z(t)||^2 = ||z_0||^2\exp\left(\int_0^t \sum_{i=1}^{n} \lambda_i\delta_i ds\right). \tag{2.7}$$

We use (1.2) to obtain

$$\sum_{i=1}^{n} \lambda_i\{A + A^* + B(t) + B^*(t)\}\delta_i \leqslant \lambda_M\{A + A^* + B(t) + B^*(t)\} \tag{2.8}$$

$$\leqslant \lambda_M(A + A^*) + \lambda_M\{B(t) + B^*(t)\}.$$

Now let $m(s)$ be the unit eigenvector of $\{B(s) + B^*(s)\}$ such that

$$\lambda_M\{B(s) + B^*(s)\} = m^*(s)\{B(s) + B^*(s)\}m(s) \tag{2.9}$$

for $0 \leqslant s \leqslant \infty$. Also, setting $B(s) = U(s) + iV(s)$ and $m(s) = \alpha(s) + i\phi(s)$, (2.9) becomes

$$\lambda_M(B(s) + B^*(s)) = \Re(m^*(s)\{B(s) + B^*(s)\}m(s)) \tag{2.10}$$

$$= \Re[\{\alpha'(s) - i\phi'(s)\}(U(s) + U'(s) + i\{V(s) - V'(s)\})(\alpha(s) + i\phi(s))]$$

$$= 2(\alpha'(s)U(s)\alpha(s) + \phi'(s)U(s)\phi(s) + \alpha'(s)\{V'(s) - V(s)\}\phi(s))$$

$$= 2\sum_{i=1}^{n} U_{ii}(s)\{\alpha_i^2(s) + \phi_i^2(s)\} + 2\sum_{i\neq j} U_{ij}(s)\{\alpha_i(s)\alpha_j(s) + \phi_i(s)\phi_j(s)\}$$

$$+ 2\sum_{i\neq j}\{V_{ji}(s) - V_{ij}(s)\}\alpha_i(s)\phi_j(s).$$

Now

$$m^*(s)m(s) = \sum_{i=1}^{n}\{\alpha_i^2(s) + \phi_i^2(s)\} = 1$$

and we obtain

$$\lambda_M\{B(s)+B^*(s)\} \leq 2\max U_{ii}(s) + 4\sum_{i\neq j}|U_{ij}(s)| + 2\sum_{i\neq j}|V_{ij}(s)-V_{ji}(s)|. \tag{2.11}$$

We conclude from (2.7), (2.8) and (2.11) that

$$||z(t)||^2 \leq ||z_0||^2 \exp\left(\omega t + 2\int_0^t \max U_{ii}(s)ds + 4\int_0^t \sum_{i\neq j}|U_{ij}(s)|ds + 2\int_0^t \sum_{i\neq j}|V_{ij}(s)-V_{ji}(s)|ds\right). \tag{2.12}$$

In case (a), by (2.3), we select $K > 0$ such that

$$||z(t)||^2 \leq K||z_0||^2 \exp\left(t\{\omega + \frac{2}{t}\int_0^t \max U_{ii}(s)ds\}\right), \tag{2.13}$$

and the result follows from (2.2). Case (b) is analogous with the use of (2.4) and (2.12).

THEOREM 2. *Consider*

$$\dot{x} = \{T + B(t)\}x \tag{2.14}$$

and assume

$$T \text{ is triangular, } T_{ij} = 0 \text{ for } j < i, \text{ and } \max \Re\{\lambda_i(T)\} = \omega, \tag{2.15}$$

$$\int_0^\infty |B_{ij}(t)|dt < \infty, \qquad i \neq j, \tag{2.16}$$

$$\limsup_{t\to\infty} \frac{1}{t}\int_0^t \max_i \Re\{B_{ii}(s)\}ds < -\omega; \tag{2.17}$$

then every solution of (2.14) *converges to* 0 *as* $t \to \infty$.

Proof. Let $x = Hy$ where

$$H = \begin{pmatrix} 1 & 0 & \dots & 0 \\ 0 & \epsilon & & \cdot \\ \cdot & & \cdot & \cdot \\ \cdot & & & 0 \\ 0 & \cdot & \cdot & \epsilon^{n-1} \end{pmatrix},$$

$\epsilon \neq 0$ then (2.14) becomes $\dot{y} = \{D + C(t)\}y$ with $D = H^{-1}TH$, $C(t) = H^{-1}B(t)H$. Proceeding as above we obtain

$$||z(t)||^2 = ||z_0||^2 \exp\left(\int_0^t \sum_{i=1}^n \lambda_i\{D + D^* + C(s) + C^*(s)\}\delta_i ds\right) \tag{2.18}$$

where z is the unitary transform of y. Now

$$(D + D^*)_{ij} = \begin{cases} 2\Re\{\lambda_i(T)\}, & i = j, \\ \epsilon^{j-i}T_{ij}, & i < j, \\ \epsilon^{i-j}\bar{T}_{ji}, & j < i, \end{cases} \tag{2.19}$$

and $C_{ij}(s) = \epsilon^{i-j}B_{ij}(s)$. By (2.16) and (2.18) there exists such a constant K that

$$(2.20)\quad ||x(t)||^2 \leq K||x_0||^2 \exp\left(t\lambda_M(D + D^*) + 2\int_0^t \max \Re\{B_{ii}(s)\}ds\right);$$

but since the eigenvalues of a matrix are continuous functions of the entries, $\lambda_M(D + D^*)$ can be made to differ arbitrarily little from $2 \max \Re(\lambda_i(T)) = 2\omega$ by choosing ϵ sufficiently small. (2.17) completes the argument.

The divergence theorems follow analogously. We omit proofs.

THEOREM 3. *Consider* (1.1). *Assume*

$$(2.21)\quad \lambda_m(A + A^*) = \omega$$

and

$$(2.22)\quad \liminf_{t\to\infty} \frac{1}{t}\int_0^t (\min \Re\{B_{ii}(s)\} - 2\sum_{i\neq j} |\Re\{B_{ij}(s)\}| - \sum_{i\neq j} |\Im\{B(s) - B'(s)\}_{ij}|)ds > -\tfrac{1}{2}\omega;$$

then every solution of (2.1) *diverges to* ∞ *as* $t \to \infty$.

THEOREM 4. *Consider* (2.14). *Assume* (2.16), T *triangular*, $\min \Re(\lambda_i(T)) = \omega$ *and*

$$(2.23)\quad \liminf_{t\to\infty} \frac{1}{t}\int_0^t \min \Re(B_{ii}(s))ds > -\omega;$$

then every solution of (2.14) *diverges to* ∞ *as* $t \to \infty$.

Theorems 2 and 4 provide a simple proof of the following familiar statement: *If*

$$\int_0^\infty ||B(t)||dt < \infty$$

and all solutions of $\dot{x} = Ax$ *either* (a) *converge to* 0 *or are bounded or* (b) *diverge to* ∞ *as* $t \to \infty$, *then the same is true of* (1.1). For (b) implies $\min \Re\{\lambda_i(A)\} > 0$. By a change of variable assume (1.1) is in the form $\dot{x} = \{T + SB(t)S^{-1}\}x$ with T triangular,

$$\int_0^\infty |\{SB(t)S^{-1}\}_{ij}|dt < \infty$$

for all (i, j) and (2.16) holds,

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t \min \Re(\{SB(s)S^{-1}\}_{ii})ds = 0,$$

and (2.23) holds. Case (b) follows by Theorem 4. Case (a) is similar.

3. The equation (1.2). If $f(t)$ is a continuous complex-valued almost-periodic (a.p.) function on $[0, \infty)$ set

$$M\{f(t)\} = \lim_{t\to\infty} \frac{1}{t}\int_0^t f(s)ds.$$

A minor modification of an argument due to Favard (2) proves the following:

THEOREM 5. *If $f(t)$ is real-valued, $M\{f(t)\} > 0$, and*

$$\int_0^t f(s)ds$$

is not bounded on $[0, \infty)$, then there exists a sequence of intervals $[a_n, b_n]$ with $b_n > a_n > 0$, $a_n < a_{n+1}$ $(n = 0, 1, \ldots)$, $\lim a_n = \infty$, such that

$$\int_{a_n}^{b_n} f(s)ds > n.$$

We show in Theorem 6 that Theorem 5 is easily applied to obtain some sufficient conditions that imply the stability of (1.2) assuming the boundedness on $[0, \infty)$ of solutions to

$$\dot{x} = A(t)x. \tag{3.1}$$

For any finite collection of a.p. functions and any $\epsilon > 0$ there exists a common relatively dense set of translation numbers with respect to ϵ. Hence we may consider $A(t)$ an a.p. matrix function.

Denote by $X(t)$ the fundamental matrix of solutions (f.m.s.) of (3.1) with

$$X(0) = I.$$

Note that

$$\limsup_{t\to\infty} ||X(t)|| < \infty \tag{3.2}$$

and

$$|X(t)| = \exp\left\{\int_0^t \operatorname{tr} A(s)ds\right\} \tag{3.3}$$

together with the Hadamard determinant theorem imply that

$$m(h) = \limsup_{t\to\infty} \int_0^t \Re \operatorname{tr} A(s+h)ds < \infty \tag{3.4}$$

for any $h > 0$. We have

THEOREM 6. *Assume*

(i) $M\{\Re \operatorname{tr} A(t)\} > 0$,

(ii) (3.2) holds,

(iii) $\limsup_{h\to\infty} m(h) < \infty$,

(iv) $\int_0^\infty ||B(s)||ds < \infty$;

then all solutions of (1.2) *are uniformly bounded on* $[0, \infty)$.

Before proceeding, note that (i) and (ii) imply $M\{\Re \operatorname{tr} A(t)\} = 0$.

Proof. First consider the translated equation

$$\dot{x} = A(t + h)x \qquad (h \geqslant 0). \tag{3.5}$$

(ii) clearly implies that all solutions of (3.5) are bounded on $[0, \infty)$ for each h.

Let $X(t; h)$ be the f.m.s. of (3.5). Suppose there exists $h_n \to \infty$ such that

$$\lim_{n\to\infty} \limsup_{t\to\infty} X(t; h_n) = \infty. \tag{3.6}$$

Then

$$\lim_{n\to\infty} \limsup_{t\to\infty} ||X(t + h_n) \operatorname{adj} X(h_n)|| \exp\left\{-\int_0^{h_n} \Re \operatorname{tr} A(s)ds\right\} = \infty,$$

and we conclude from (ii) that

$$\int_0^t \Re \operatorname{tr} A(s)ds$$

is not bounded on $[0, \infty)$. By Theorem 5 there exists a sequence of intervals $[a_n, b_n]$ such that

$$\int_{a_n}^{b_n} \Re \operatorname{tr} A(s)ds \geqslant n. \tag{3.7}$$

Setting $l_n = b_n - a_n$ and $s = a_n + t$, (3.7) becomes

$$\int_0^{l_n} \Re \operatorname{tr} A(t + a_n)dt \geqslant n,$$

and we conclude that $m(a_n) \geqslant n$, contradicting (iii). Hence there exists $K > 0$ such that

$$\limsup_{h\to\infty} \limsup_{t\to\infty} X(t; h) = K < \infty. \tag{3.8}$$

Let $u(b, t)$ be a solution of (1.2) with $u(b, 0) = b$. Using the variation of parameters formula and taking norms on both sides, we have

$$||u(b, t)|| \leqslant ||X(t)||\,||b|| + \int_0^t ||X(t)X^{-1}(s)||\,||B(s)||\,||u(b, s)||ds. \tag{3.9}$$

In (3.9) $t \geqslant s$, $t - s = h \geqslant 0$,

$$\frac{d}{ds}\{X(s + h)X^{-1}(h)\} = A(s + h)X(s + h)X^{-1}(h)$$

and it is obvious that $X(s + h)X^{-1}(h)$ is the f.m.s. of (3.5). By (3.8) we conclude that

$$\limsup_{t\to\infty} ||X(t)X^{-1}(s)|| = K < \infty.$$

Using an inequality due to Gronwall **(3)**, we have

$$||u(b, t)|| \leqslant K||b|| \exp\left(K \int_0^t ||B(s)||ds\right)$$

and (iv) completes the proof.

We may remark that the argument applied to (3.5) will yield the usual stability theorem in case $A(t)$ is purely periodic without use of the Floquet representation of the f.m.s. as a product of exponential and periodic matrix functions.

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ON PARTIAL DIFFERENTIAL EQUATIONS IN A FIELD OF PRIME CHARACTERISTIC

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In classical analysis ordinary differential equations and partial differential equations are distinct concepts, and the transition from one derivation to several partial derivations changes some of their properties distinctly. On the other hand, the algebraic theories of modified ordinary and partial differential equations **(5; 6)**, based on the differentiations in the sense of Hasse **(2)** and Schmidt **(3)** and the multidifferentiations in the sense of Jaeger **(4)**, turn out to be strikingly similar in the case of fields of prime number characteristic. However, the differential equations in fields of prime number characteristic in the usual, unmodified sense are special cases of the modified ones because of the relations between their respective operators; hence this similarity must also appear in the unmodified case. In the following, an easy explanation for this similarity is given, provided that only derivations of a separably generated algebraic function field F in n independent indeterminates over its ground field K are considered. For in this case it is shown that all partial differential equations can be replaced by ordinary differential equations in a suitable derivation D of F over K. This D is simply constructed in such a way that the set

$$(D, D^p, D^{p^2}, \ldots, D^{p^{n-1}})$$

is a basis for the F-module of all derivations of F over K. An explicit example of such a "replacing derivation" is given in this paper.

Let F be a separably generated algebraic function field of n independent indeterminates of prime number characteristic $p > 0$ with a separating transcendence basis $x = (x_1, x_2, \ldots, x_n)$ over its ground field K. Let D_i $(i = 1, 2, \ldots, n)$ be the partial derivations of F over K uniquely defined **(1)** by $D_i(x_j) = \delta_{ij}$ where δ_{ij} denotes the Kronecker delta. The linear mappings D_i of F into itself are commutative under multiplication; the abbreviation

$$D^\alpha = \prod_{i=1}^{n} D_i^{\alpha_i}$$

will be used for their general products where $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_n)$ is a vector whose components are non-negative integers and

$$D_i^0 \qquad (i = 1, 2, \ldots, n)$$

denotes the identity mapping 1. The multisequence $\{D^\alpha\}$ is called a *multidifferentiation of dimension* n; if $n = 1$ it is also called a *differentiation.*

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The formal expressions $A = \sum_\alpha a_\alpha D^\alpha$ ($a_\alpha \in F$) where the sum is extended over finitely many distinct vectors α whose components are non-negative integers are called *multidifferential operators in D* (or also *differential operators* if $n = 1$). They define mappings

$$y \to A(y) = \sum_\alpha a_\alpha D^\alpha(y)$$

of F into itself and form a ring $\Omega = \Omega(D, F)$ of mappings when addition is defined in the usual way and multiplication by $D_i y = D_i(y)\, D_i^0 + y D_i$ ($y \in F$; $i = 1, 2, \ldots, n$). The zero element of this ring is denoted by 0.

LEMMA 1. *The operators*

$$D_i^q \qquad (i = 1, 2, \ldots, n)$$

where $q = p^t$ and t is any positive integer, map every element of F onto zero.

Proof. The operator defines a derivation of F over K. It has the property that

$$D_i^q(x_j) = 0 \qquad (j = 1, 2, \ldots, n);$$

hence, its restriction to $K[x]$ is the trivial derivation (i.e. the zero operator). But the prolongation of a derivation of $K[x]$ to F is unique **(1)**, and hence this prolongation is a zero operator.

Let P be the two-sided ideal of Ω generated by

$$D_1^p, D_2^p, \ldots, D_n^p.$$

If $a_\alpha \neq 0$ and $D^\alpha \notin P$ the number $\sum \alpha_i$ is called the *order of* $a_\alpha D^\alpha$. Let $A = \sum_\alpha a_\alpha D^\alpha$ be a multidifferential operator which does not lie in P, then the minimum of the orders of the additive terms $a_\alpha D^\alpha$ for which $a_\alpha \neq 0$ and $D^\alpha \notin P$ holds is called the *minimal order m_A of A*. If all terms $a_\alpha D^\alpha$ of A with $a_\alpha \neq 0$ have the same order the operator A is called *homogenous of order m_A*. Order and minimal order are not defined for elements of P.

LEMMA 2. *No multidifferential operator $A = \sum_\alpha a_\alpha D^\alpha$ of minimal order $m_A > 1$ defines a derivation.*

Proof. We have obviously $A(x_i) = 0$ ($i = 1, 2, \ldots, n$). Suppose now that A is a derivation, then it must follow that also $A(x^\mu) = 0$ where

$$x^\mu = \prod_{i=1}^{n} x_i^{\mu_i}, \qquad \mu = (\mu_1, \mu_2, \ldots, \mu_n)$$

can be chosen such that

$$\sum_{i=1}^{n} \mu_i = m_A,$$

all $\mu_i < p$ and $a_\mu \neq 0$. But it can be seen easily that

$$D^\alpha(x^\mu) \begin{cases} \neq 0 \text{ if } & \alpha = \mu \\ = 0 \text{ if } & \sum \alpha_i = m_A, \text{ but } \alpha \neq \mu \\ = 0 \text{ if } & \sum \alpha_i > m_A. \end{cases}$$

Hence we have $A(x^\mu) = \sum_\alpha a_\alpha D^\alpha(x^\mu) \neq 0$, a contradiction.

COROLLARY 1. *Let H be the set of all multidifferential operators of minimal orders greater than* 1. *Then $I = H \cup P$ is a left-ideal of* Ω.

COROLLARY 2. *A multidifferential operator A is a zero operator if and only if $A \in P$.*

Since two multidifferential operators A and B define the same mapping of F into itself if and only if $A \equiv B \pmod{P}$, it is sufficient to consider the operator ring Ω/P. A multidifferential operator $A = \sum_\alpha a_\alpha D^\alpha$ is called *reduced with respect to* P if none of its non-zero terms $a_\alpha D^\alpha$ lies in P; the reduced operators can be taken as representatives of the residue classes of Ω/P. Furthermore we can identify the additive group, consisting of the operator 0 and all homogenous multidifferential operators of order 1, with the F-module $M = M(D, F)$ of all derivations of F over K.

COROLLARY 3. *If $A^q \equiv A_1 \pmod{I}$ holds with $q = p^t$ for some positive integer t and $A_1 \in M$, then $A^q \equiv A_1 \pmod{P}$.*

Proof. A^q and A_1 and hence, $A^q - A_1$ define derivations. Since all derivations induced by elements of I are trivial it follows that $A^q - A_1 \in P$.

THEOREM. *There exists a differentiation d of F over K such that for each multidifferential operator A of $\Omega(D, F)$ there exists a differential operator $\bar{A}$ of $\Omega(d, F)$ which defines the same mapping of F into itself as A.*

Proof. It is sufficient to show the existence of a basis $d = (d_1, d_2, \ldots, d_n)$ of the F-module M such that $d^q \equiv d_{i+1} \pmod{P}$ holds for $q = p^i, i = 1, 2, \ldots, n-1$.

Example. Take $d = \sum x_i^{p+1} D_i$, then it follows that

$$d^k \equiv \sum_{i=1}^{n} x_i^{kp+1} D_i \pmod{I}$$

(k any positive integer), and, by using Corollary 3, we have especially

$$d^q \equiv \sum_{i=1}^{n} x_i^{qp+1} D_i \pmod{P}.$$

But the elements

$$d_k = \sum_{i=1}^{n} x_i^{p^k+1} D_i \qquad k = 1, 2, \ldots, n$$

form a basis of the F-module M since

$$\det(x_i^{p^k+1})_{i,k=1,2,\ldots,n} \not\equiv 0.$$

COROLLARY. *Every partial differential equation in F over K can be written as an ordinary differential equation in F over K.*

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COUNTABLE PARACOMPACTNESS AND SOUSLIN'S PROBLEM

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1. Introduction. A linearly ordered space S in which neighborhoods are segments is called a Souslin space if

(i) S is not separable, but

(ii) every collection of disjoint segments of S is countable.

Whether a Souslin space exists is not known; this is the problem referred to in the title and was proposed by Souslin in **(2)**.

A *covering* of a topological space T is a collection of open subsets of T whose sum is T. A covering is *locally finite* if every point of T is in some open set which intersects only a finite number of sets of the covering. A topological space T is said to be *countably paracompact* if every countable covering of T has a locally finite refinement **(1)**.

A topological space T is *normal* if for every two disjoint closed sets K_1 and K_2 of T there are disjoint open sets H_1 and H_2 of T containing K_1 and K_2, respectively; if, in addition, every point of T is a closed set, then T is a *normal Hausdorff space*. It is not known whether every normal Hausdorff space is countably paracompact.

In this paper the following will be proved:

THEOREM. *If there exists a Souslin space, then there exists a normal Hausdorff space which is not countably paracompact.*

Let us say that a topological space has property D if the following is true: *whenever $C_1, C_2, C_3, \ldots$ is a decreasing sequence of closed sets having no common part, then there exists a sequence $D_1, D_2, D_3, \ldots$ of open sets having no common part such that, for each n, D_n covers C_n.*

In the proof of our theorem we shall use the following result due to Dowker **(1**, p. 220**)**: *A normal space is countably paracompact if and only if it has property* D.

2. Preliminary constructions and notation. We now assume that there is a Souslin space S. If S contains any separable segments, the collection of all maximal separable segments is at most countable; and the sum of this collection of segments is not dense in S, since S would otherwise be separable. Hence *there is a segment s in S none of whose subsegments is separable.*

Let R_1 be a collection of disjoint subsegments of s. Suppose R_α has been defined for all ordinals $\alpha < \beta$, where β is a countable ordinal. If β is not a limit

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ordinal, let S_β be the sum of the segments of $R_{\beta-1}$ and E_β be a set consisting of one point from each term of $R_{\beta-1}$; then let R_β be the set of all maximal segments of $S_\beta - E_\beta$. If β is a limit ordinal, let S_β be the set of all points which belong to some segment of R_α for every $\alpha < \beta$; and let R_β be the collection of all maximal segments in S_β. Since each E_α is countable for all non-limit ordinals $\alpha < \beta$, the closure of the sum of all such E_αs is separable; hence R_β exists for all β.

Let R be the sum of all R_β where β is a countable ordinal.

For each term x of R let $\phi(x)$ denote the ordinal such that x belongs to $R_{\phi(x)}$.

If β is a countable limit ordinal, then for each x in R_β we consider a sequence $f_1(x), f_2(x), f_3(x), \ldots$ of segments in R_β such that

(a) $f_i(x_1) = f_j(x_2)$ if and only if $i = j$ and $x_1 = x_2$.

(b) if x is a subsegment of a segment y of R where $\phi(y) < \beta$, then, for some n, $f_i(x)$ is a subsegment of y whenever $i > n$.

From now on we will assume that whenever one of the letters x, y, z, or w is used it stands for a segment of R; whenever one of the letters i, j, m, or n is used it stands for a positive integer; and whenever a greek letter is used it stands for a countable ordinal.

3. Construction of T. The points of T are the ordered pairs (x, n) where x is in R and n is a positive integer. We now define a neighborhood system in T.

3.1. If $\phi(x)$ is not a limit ordinal, a neighborhood of (x, n) consists of the point (x, n) alone.

3.2. If $\phi(x)$ is a limit ordinal and $\beta < \phi(x)$, a neighborhood of $(x, 1)$ consists of all points $(y, 1)$ such that x is a subsegment of y and $\beta < \phi(y) \leqslant \phi(x)$.

3.3. To define neighborhoods of $p = (x, n)$ if $\phi(x) = \beta$ is a limit ordinal and $n > 1$, we proceed inductively, assuming that neighborhoods have been defined for all (y, m) with $m < n$ and for all (y, n) with $\phi(y) < \beta$.

Let $F_i(p)$, for each i, be the set of all points $(f_j(x), n - 1)$ where $j > i$.

Let $G_\alpha(p)$, for each α, be the set of all points (y, n) such that x is a subsegment of y and $\alpha < \phi(y) < \beta$.

A set N is a neighborhood of $p = (x, n)$ if N is the sum of

(a) a neighborhood of each point of some $F_i(p)$,

(b) a neighborhood of each point of some $G_\alpha(p)$, and

(c) the point p itself.

3.4. For better orientation and later use we mention the following facts about T.

(i) The set C_n consisting of all (x, m) with $m \geqslant n$ is closed.

(ii) For any β, the set of all (x, n) with $\phi(x) \leqslant \beta$ is countable, open, and closed.

(iii) The neighborhoods of points of the form $(x, 1)$ as described in 3.2 are open and closed.

(iv) If $\phi(x)$ is a limit ordinal, $n > 1$, and x is a subsegment of z, then every neighborhood of (x, n) contains points of the form (y, m), for any $m < n$, such that $\phi(y) = \phi(x)$ and y is a subsegment of z.

4. To prove that T has the desired properties we introduce more machinery.

Let Q be a subcollection of R. If x has the property that every subsegment y of x (in R) contains a segment z such that z is in Q, then we say that x *is Q-full.* If x is Q-full, then, for any β, we let $L(x, \beta, Q)$ be the collection of all y such that

(a) y is a subsegment of x belonging to Q and $\phi(y) \geqslant \beta$;

(b) y is not a proper subsegment of any segment of R for which (a) holds.

Since the segments of $L(x, \beta, Q)$ are disjoint, they are countable, and there exists a smallest ordinal $\delta(x, \beta, Q)$ such that $\phi(y) < \delta(x, \beta, Q)$ for every y in $L(x, \beta, Q)$.

We note that $\beta < \delta(x, \beta, Q)$.

5. Proof that T does not have property D. We let C_n be the closed set consisting of all points (x, m) where $m \geqslant n$. Suppose that, for each n, D_n is an open set containing C_n.

LEMMA 5.1. *Let n be fixed. For every x there exists a subsegment y such that, if z is any subsegment of y, $(z, 1)$ is a point of D_n.*

Let Q_n be the set of all x for which $(x, 1)$ is not in D_n. The lemma is equivalent to the assertion: no x is Q_n-full.

To prove the Lemma, suppose x is Q_n-full. Let $\beta_1 = \phi(x)$, $\beta_i = \delta(x, \beta_{i-1}, Q_n)$ for $i > 1$, and β be the limit ordinal of $\beta_1, \beta_2, \beta_3, \ldots$. By 3.3, there is a segment y of R_β such that y is a subsegment of x and $(y, 1)$ is in D_n.

For each i, let z_i be the segment of $L(x, \beta_i, Q_n)$ of which y is a subsegment. Since z_i is in Q_n, $(z_i, 1)$ is not in D_n. By 3.2, $(y, 1)$ is a limit point of the sequence $(z_1, 1), (z_2, 1), (z_3, 1), \ldots$. Consequently $(y, 1)$ is not in D_n, and this contradiction proves Lemma 5.1.

5.2. Let P_n be the set of all x such that $(y, 1)$ is in D_n if y is a subsegment of x. Lemma 5.1 shows that every x is P_n-full for every n.

Let x be a segment of R_1; pick γ such that $\delta(x, 1, P_n) < \gamma$ for every n. Let y be a subsegment of x belonging to R_γ. We see that $(y, 1)$ is in D_n for every n.

Consequently $D_1 \cdot D_2 \cdot D_3 \cdot \ldots$ exists and T does not have property D.

6. Proof that T is a normal Hausdorff space. It is clear that every point of T is a closed set.

Let H and K be disjoint closed subsets of T. For each n let H_n and K_n be the sets of all x for which (x, n) is in H or K, respectively.

LEMMA 6.1. *Suppose i and j are integers and x and y are segments of R where y is a subsegment of x. Then, if x is H_i-full, y is not K_j-full.*

Suppose, on the contrary, that x is H_i-full and y is K_j-full. Choose $\gamma_0 > \phi(y)$. For odd n, let $\gamma_n = \delta(y, \gamma_{n-1}, K_j)$; if n is even, let $\gamma_n = \delta(x, \gamma_{n-1}, H_i)$. Let γ be the limit ordinal of $\gamma_1, \gamma_2, \gamma_3, \ldots$.

There exists a subsegment z of y in R_γ. If $i = j$ the point (z, j) is clearly a limit point of both H and K which is impossible since H and K are closed and disjoint. Assume $i < j$. Then clearly (z, j) is a limit point of K. But every neighborhood of (z, j) contains points of the form (w, i) where w is a subsegment of y in R_γ; and (w, i) is a limit point of H. So (z, j) is also a limit point of H which is a contradiction.

The case $j < i$ is treated similarly.

LEMMA 6.2. *There is an ordinal μ with the following property: no x in R_μ contains two segments y and z such that y is in H_n and z is in K_m, for any choice of m and n.*

Let P_1 be the set of all x none of whose subsegments in R belongs to H_n for any n. Let P_2 be the set of all x none of whose subsegments in R belongs to K_m for any m. Let P_i', for i equal to 1 or 2, be the collection of all maximal segments in P_i. Then P_i' is countable, and we can choose μ so that $\phi(y) < \mu$ for all y in P_i'. Choose x in R_μ.

Case 1: *No subsegment of x is K_m-full for any m.*

Then every segment is I_m-full where I_m is the set of all segments none of whose subsegments belongs to K_m. Put $\alpha_0 = \mu$, $\alpha_m = \delta(x, \alpha_{m-1}, I_m)$, and let α be the limit ordinal of $\alpha_1, \alpha_2, \alpha_3, \ldots$.

If z is a subsegment of x in R_α, then z is in P_2; so z is a subsegment of w for some w in P_2'. By our choice of μ, we see that x is a subsegment of w, and it follows that no subsegment of x in R belongs to K_m for any m.

Case 2: *Some subsegment y of x is K_m-full for some m.*

By 6.1 no subsegment of y is then H_n-full for any n. Proceeding as in Case 1, we see that y is a subsegment of a segment w of P_1', so that x is also a subsegment of w, and no subsegment of x belongs to H_n for any n.

This completes the proof of Lemma 6.2.

6.3. Choosing μ in accordance with Lemma 6.2, we let X be the set of all (x, n) for which $\phi(x) < \mu$. Then X is a countable, closed and open subset of T; and we can order the points of X in a simple countable sequence $p_1, p_2, p_3, \ldots$.

Let $A_0 = H \cdot X$ and $B_0 = K \cdot X$. We shall construct nonintersecting sequences $A_1, A_2, A_3, \ldots$ and $B_1, B_2, B_3, \ldots$ of closed sets. Having constructed A_{m-1} and B_{m-1}, consider the point $p_m = (x, n)$.

I. *Suppose that p_m is not in B_{m-1}.*

Case 1. If $\phi(x)$ is not a limit ordinal, let $A_m = A_{m-1} + p_m$, and let $B_m = B_{m-1}$.

Case 2. If $\phi(x)$ is a limit ordinal and $n = 1$, let J be a neighborhood of p_m of the type described in 3.2 such that J does not intersect B_m. Put $A_m = A_{m-1} + J$ and $B_m = B_{m-1}$.

Case 3. If $\phi(x)$ is a limit ordinal and $n > 1$, choose i and β so that no point of $F_i(p_m)$ or of $G_\beta(p_m)$ (using the notation introduced in 3.3) is a point of A_{m-1}. We put $A_m = A_{m-1} + p_m + F_i(p_m) + G_\beta(p_m)$ and $B_m = B_{m-1}$.

II. *If p_m is in B_{m-1}, then perform the operations of cases* 1, 2, *and* 3 *above interchanging A and B.*

Then A_m and B_m are closed and disjoint and the induction is complete. We now prove

6.4. *The sets $A = A_1 + A_2 + A_3 + \ldots$ and $B = B_1 + B_2 + B_3 + \ldots$ are open.*

Proof. If $p = (x, n)$ is in A, then p is certainly an interior point of A whenever $\phi(x)$ is not a limit ordinal and whenever $n = 1$. Suppose A is not open. Then there is a point $(x, n) = p$ of A which is not an interior point of A, but such that every point (y, i) of A where $i < n$ and every point (z, n) where $\phi(z) < \phi(x)$ is an interior point of A. If m is the integer such that $p_m = p$, the above construction of A_m and the definition of neighborhood in 3.3 show that a neighborhood of p_m is included in A.

Hence A is open and the same is, of course, true of B.

6.5. Let V' and W' be the collections of all x in R_α which have a subsegment in some H_i or in some K_j, respectively. Let V and W be the sets of all points (y, n) with y a proper subsegment of a segment of V' or W', respectively.

Then V and W are open disjoint subsets of T by Lemma 6.2.

Finally, the sets $A + V$ and $B + W$ are disjoint and open and cover H and K, respectively. Hence T is normal and the proof is complete.

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A NOTE ON THE SEPARABILITY OF AN ORDERED SPACE

B. J. BALL

An *open interval* of a simply ordered set S is a subset I of S such that either
(1) for some $a \in S$, $I = \{x \in S | x < a\}$,
(2) for some $a \in S$, $I = \{x \in S | a < x\}$, or
(3) for some $a \in S$ and $b \in S$, $I = \{x \in S | a < x < b\}$.

A simply ordered set with its interval topology (i.e., the topology in which "neighborhood of x" means "open interval containing x") will be called an *ordered space.*

It is shown that a connected ordered space S is separable provided it satisfies Souslin's condition **(2)** (i.e., there exists no uncountable collection of mutually exclusive open subsets of S) and there is a countable family F of continuous functions of S into itself such that each point p of S is a limit point of $\{f(p) | f \in F\}$. If S is not assumed to satisfy Souslin's condition, the existence of such a family F does not imply that S is separable; however, if no element of F has a fixed point or if the elements of F can be arranged in a sequence $\{f_n\}$ such that for each point p of S, $\{f_n(p)\} \to p$, then S must satisfy Souslin's condition and hence must be separable.

Notation. If S is an ordered space and a and b are elements of S such that $a < b$, then ab will denote the open interval of S with end points a and b; i.e., $ab = \{x \in S | a < x < b\}$. As usual, $S \times S$ will denote the topological product of S with itself and if f is a function of S into itself, then $G(f)$ will denote the "graph" of f in $S \times S$; that is, $G(f) = \{(x, f(x)) | x \in S\}$.

THEOREM 1. *Suppose S is a connected ordered space and F is a countable family of continuous functions of S into itself such that each point p of S is a limit point of $\{f(p) | f \in F\}$. If S satisfies Souslin's condition, then S is separable.*

LEMMA. *Under the above hypothesis, if a, b and c are elements of S such that $a < b < c$, then for some f in F, $G(f)$ intersects the subset $(ab \times bc) + (bc \times ab)$ of $S \times S$.*

Proof of Lemma. Since b is a limit of $\{f(b) | f \in F\}$, there exists an element f of F such that $f(b) \in ac$ and $f(b) \neq b$. Suppose $f(b) \in bc$. Since f is continuous, there exists a neighborhood V of b such that $f(V) \subset bc$. Since S is connected, there exists a point x of V such that $x \in ab$. Since $x \in ab$ and $f(x) \in bc$, $(x, f(x)) \in ab \times bc$. Similarly, if $f(b) \in ab$, there exists an x in bc such that $(x, f(x)) \in bc \times ab$. Hence $G(f)$ intersects $(ab \times bc) + (bc \times ab)$.

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Proof of Theorem 1. In **(1)** it is shown that if S is not separable there exists an uncountable collection $\mathfrak{U}$ of mutually exclusive open subsets of $S \times S$ such that if $U \in \mathfrak{U}$, there exist points a, b, c of S such that $a < b < c$ and $U = (ab \times bc) + (bc \times ab)$. Since F is countable and for each U in $\mathfrak{U}$ there is an f in F such that $G(f)$ intersects U, there exists an f_0 in F such that $G(f_0)$ intersects each of uncountably many elements of $\mathfrak{U}$. Since for each U in $\mathfrak{U}$, the intersection of $G(f_0)$ and U is an open subset of $G(f_0)$, $G(f_0)$ contains uncountably many mutually exclusive open sets. But since f_0 is continuous, $G(f_0)$ is homeomorphic to S and hence satisfies Souslin's condition.

That Theorem 1 does not remain true if the requirement that S satisfy Souslin's condition be dropped is shown by the following example.

Let the points of S be the ordered pairs (x, y) of real numbers such that $0 \leqslant y \leqslant 1$ and let (x_1, y_1) precede (x_2, y_2) in S if and only if either $x_1 < x_2$ or $x_1 = x_2$ and $y_1 < y_2$. Then S is a connected ordered space but is not separable since it does not satisfy Souslin's condition. For each positive integer n, let

$$g_n(x, y) = \left(x, y - \frac{y}{n} + \frac{y^2}{n}\right), \quad h_n(x, y) = \left(x + \frac{(-1)^n}{n}, y\right).$$

Then for each n, both g_n and h_n are continuous functions of S into itself. If $p = (x, y)$ then p is a limit point of $\{g_n(p)\}$ if $0 < y < 1$ and p is a limit point of $\{h_n(p)\}$ if $y = 0$ or $y = 1$. Hence if $F = \{\{g_n\} + \{h_n\}\}$, then for each p in S, p is a limit point of $\{f(p) | f \in F\}$.

THEOREM 2. *If S is a connected ordered space and F is a countable family of continuous functions of S into itself such that* (1) *no element of F has a fixed point and* (2) *each point p of S is a limit point of $\{f(p) | f \in F\}$, then S is separable.*

LEMMA 1. *Every uncountable subset of a connected ordered space has a limit point.*

Proof of Lemma 1. Suppose S is a connected ordered space and M is an uncountable subset of S which has no limit point. Let a be a point of S and suppose there are uncountably many points x of M such that $a < x$. Since S is connected, every infinite bounded subset of M has a limit point. Hence if b is a point of S such that $a < b$, then ab contains not more than a finite number of points of M. It follows that there exists a sequence $\{x_n\}$ of points of M such that for each n, $a < x_n < x_{n+1}$. Since for each n there are not more than a finite number of points of M in the interval ax_n, but there are uncountably many points x of M such that $a < x$, the sequence $\{x_n\}$ is bounded and hence has a limit point.

LEMMA 2. *If S is a connected ordered space and G is an uncountable collection of mutually exclusive intervals of S, then there exist a point p of S and an infinite countable subcollection G' of G such that every neighborhood of p contains all but a finite number of the elements of G'.*

Proof of Lemma 2. Let M denote a set consisting of one and only one point of each element of G. Since M is uncountable, it has a limit point. If q is a limit point of M, then either every open interval containing q contains a point x of M such that $x < q$ or every open interval containing q contains a point x of M such that $q < x$. Hence there exists a sequence $\{x_n\}$ of points of M such that either (1) for each n, $x_n < x_{n+1} < q$ or (2) for each n, $q < x_{n+1} < x_n$. Since S is connected and $\{x_n\}$ is bounded, $\{x_n\}$ has both a greatest lower bound in S and a least upper bound in S. In case (1), let p be the greatest lower bound of $\{x_n\}$ and in case (2), let p be the least upper bound of $\{x_n\}$. In either case it is clear that the sequence $\{x_n\}$ converges to p. For each n, let g_n denote the element of G which contains x_n. It is easily seen that since the elements of G are mutually exclusive, every neighborhood of p contains all but a finite number of the intervals $g_1, g_2, g_3, \ldots$.

Proof of Theorem 2. Suppose G is an uncountable collection of mutually exclusive open intervals of S. For each element g of G, there exists an element f_g of F such that $f_g(g)$ intersects g. Hence there exist an element f of F and an uncountable subcollection G' of G such that for each element g of G', $f(g)$ intersects g. From Lemma 2 it follows that there exist a point p of S and a sequence $g_1, g_2, g_3, \ldots$ of elements of G' such that every neighborhood of p contains all but a finite number of the intervals $g_1, g_2, g_3, \ldots$. For each n, let p_n be a point of g_n such that $f(p_n) \in g_n$. Then $\{p_n\} \to p$ and hence since f is continuous, $\{f(p_n)\} \to f(p)$. But since for each n, $f(p_n) \in g_n$, $\{f(p_n)\} \to p$. Hence $f(p) = p$ and f has a fixed point. Hence S satisfies Souslin's condition. It follows from Theorem 1 that S is separable.

NOTE. If in the hypothesis of Theorem 2 the elements of F are required to be homeomorphisms of S onto itself, it can be shown by a direct argument that for each point p of S the set

$$\{f^n(p) | f \in F,\ n = 0, \pm 1, \pm 2, \ldots\}$$

is a countable dense subset of S.

THEOREM 3. *If S is a connected ordered space and $\{f_n\}$ is a sequence of continuous functions of S into itself such that for each point p of S, $\{f_n(p)\} \to p$ and for infinitely many integers n, $f_n(p) \neq p$, then S is separable.*

Proof. Suppose G is an uncountable collection of mutually exclusive open intervals of S. For each element g of G there exist a point p_g of g and an integer n_g such that for $n \geq n_g$, $f_n(p_g) \in g$. Hence there exist an integer n and an uncountable subcollection G' of G such that for each element g of G', $n = n_g$. By Lemma 2 to Theorem 2, there exist a point p of S and a sequence $g_1, g_2, g_3, \ldots$ of elements of G' such that every neighborhood of p contains all but a finite number of the intervals $g_1, g_2, g_3, \ldots$. It follows as in the proof of Theorem 2 that for $k \geq n$, $f_k(p) = p$. But this is impossible by hypothesis. Hence S satisfies Souslin's condition. Hence, by Theorem 1, S is separable.

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UNE CARACTERISATION VECTORIELLE-METRIQUE DES ESPACES L^1

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Soit C un espace de Banach (nous ne parlerons que d'espaces sur le corps des nombres réels, pour simplifier). On dit que C possède la *propriété de Nachbin* ou est un espace (N), si pour toute application linéaire continue u dans C d'un sous-espace vectoriel F d'un espace de Banach E, il existe une application linéaire de même norme de E dans C qui prolonge u. Une étude complète de ces espaces a été faite par Nachbin **(9)**, qui prouve: pour tout espace compact stonien[1] K, l'espace $C(K)$ des fonctions continues sur K, muni de sa norme uniforme, est un espace (N); et réciproquement si C est un espace (N) et si sa boule unité admet un point extrémal, alors il est isomorphe (avec sa norme) à un espace $C(K)$ construit sur un espace compact stonien K convenable. C'est surtout la réciproque qui est profonde. Quant à la partie directe, c'est une conséquence facile du résultat suivant plus fort (dont je m'excuse de ne pas me rappeler une référence précise): Si K est un espace compact stonien, et si $C(K)$ est une sous-algèbre d'une algèbre $C(L)$ (L compact quelconque), il existe une projection de norme 1 de $C(L)$ sur $C(K)$ qui est une *représentation d'algèbre*[2]; résultat parfois commode, et qui réduit dans une certaine mesure les propriétés algébrico-topologiques de $C(K)$ à celles d'une algèbre $l^\infty(I)$ formée de toutes les fonctions bornées sur un ensemble d'indices I (car prenant $I = K$, $C(K)$ se trouve plongé dans $l^\infty(I)$ isomorphe à $C(\hat{I})$ ($\hat{I}$ compactifié de Stone de I), et on peut appliquer le résultat précédent). Enfin, dans le cas particulier d'un espace L^∞ construit sur une mesure arbitraire qui est isomorphe

Reçu le 27 septembre, 1954. Cet article est destiné surtout à donner une preuve correcte d'un des résultats annoncés dans une ancienne note **(3**, théorème 5**)**. (Je m'étais aperçu peu après la publication de cette note que la preuve de ce résultat était incomplète.) En plus (en attendant la publication prochaine de **(4)**), le lecteur pourra consulter **(6)** pour ce qui concerne les produits tensoriels topologiques.

[1]Rappelons qu'on appelle espace *stonien* un espace compact K tel que l'espace des fonctions réelles continues sur K soit complètement réticulé pour sa relation d'ordre naturelle.

[2]On peut donner de cet énoncé diverses variantes, telle que: soit u une représentation d'une algèbre $C(L)$ dans $C(K)$, supposons $C(L)$ plongée dans une algèbre $C(L')$, alors u se prolonge en un *homomorphisme d'algèbre* de $C(L')$ dans $C(K)$ (L, L', K sont des compacts, K stonien.) C'est vrai en effet si $C(K)$ est du type $l^\infty(I)$, algèbre des fonctions bornées sur un ensemble d'indices I, i.e. si K est le compactifié de Stone d'un ensemble discret I; car alors on est ramené à prolonger un caractère défini sur $C(L)$, ce qui est classique. Et d'autre part, on se ramène au cas particulier précédent, grâce à la remarque du texte que $C(K)$ est plongé dans $l^\infty(I)$, quand I désigne K muni de la topologie discrète, et qu'il existe alors une projection de $l^\infty(I)$ sur $C(K)$ qui est une *représentation*. Le théorème de prolongement de Nachbin peut se démontrer de la même façon en se ramenant au cas d'un espace $l^\infty(I)$, et de là au problème de prolongement d'une *forme* linéaire continue, lequel est résolu par le théorème de Hahn-Banach.

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à un espace $C(K)$, avec K compact stonien, en vertu d'un classique théorème de Gelfand-Stone, une démonstration très naturelle de la partie directe du théorème de Nachbin se présente dans la théorie des produits tensoriels topologiques, où on obtient très simplement toutes les propriétés vectorielles-métriques essentielles des espaces $C(K)$ (K compact quelconque) et L^1, propriétés qui ne sont que des variantes diverses d'une seule quelconque entre elles, *p . ex.* celle de Nachbin (**4**, §2, N° 2).

Nous étudions ici une propriété duale de la propriété (N), qui nous conduira vers les espaces L^1 (au lieu des espaces L^∞). Soit u une application d'un espace de Banach L dans le quotient E/F d'un espace de Banach E par un sous-espace vectoriel fermé F, on dit qu'un application v de L dans E *relève* u, si u est le composé de v et de l'application canonique de E sur E/F. Considérons E/F comme un sous-espace de son bidual $(E/F)''$, lequel s'identifie, comme on sait, au quotient E''/F'' du bidual de E par le bidual de F (lui même identifié à un sous-espace de E'', par la bitransposée de l'application identique de F dans E). Alors une application u de L dans E/F est aussi une application de L dans le quotient E''/F'', et la définition précédente donne un sens à l'assertion: l'application v de L dans E'' relève l'application u de L dans E/F. Ceci posé, on a l'immédiate

PROPOSITION 1. *Soit L un espace de Banach. Les conditions suivantes sur L sont équivalentes:*

(1) *Quel que soit l'espace de Banach E et le sous-espace vectoriel fermé F de E, l'application canonique*

$$L \hat{\otimes} F \to L \hat{\otimes} E$$

(4, §1, *No* 1 *et No* 3**)** *est un isomorphisme métrique du premier espace dans le second.*

(2) *Le dual L' possède la propriété de Nachbin.*

(3) *Quel que soit l'espace de Banach E et le sous-espace faiblement fermé F° de son dual E' (F° polaire d'un sous-espace F de E), toute application linéaire continue u de L dans E'/F° se relève en une application linéaire de même norme de L dans E'.*

(4) *Quel que soit l'espace de Banach E et le sous-espace vectoriel fermé F de E, toute application linéaire continue u de L dans E/F se relève en une application linéaire de même norme de L dans E''.*

Démonstration. (1) signifie aussi que toute forme linéaire continue sur $L \hat{\otimes} F$ provient d'une forme de norme égale sur $L \hat{\otimes} E$, ou encore (en interprétant ces formes linéaires comme des formes bilinéaires) que toute forme bilinéaire sur $L \times F$ se prolonge en une forme bilinéaire de même norme sur $L \times E$. Interprétant les formes bilinéaires sur $L \times F$ comme des applications linéaires de F dans L', resp. de L dans $F' = E'/F^\circ$, on trouve que (1) équivaut à (2) resp. à (3). (3) implique (4) puisque pour donner un sens à (4), il faut

par définition regarder u comme une application linéaire de L dans E''/F'', or F'' est un sous-espace vectoriel faiblement fermé du dual E'' de E' (savoir le polaire dans E'' de $F^\circ \subset E'$), et en vertu de (3) il existe bien une application linéaire de L dans E'', de même norme que u, relevant u. Enfin prouvons que (4) implique (3). Avec les notations de (3), il existera en effet, en vertu de la condition (4), une application linéaire v de norme $||u||$ de L dans le bidual E''' de E' qui relève u, i.e. telle que pour $x \in L$, la forme vx sur E'' ait pour restriction à F'' l'élément ux de E'/F° (identifié au dual de F). Désignant par wx la restriction de vx à E, w est une application linéaire de L dans E' qui est le relèvement cherché.

Remarque 1. Il est facile de s'assurer que dans l'énoncé des conditions précédentes, il suffit de supposer que les espaces E, F sont de dimension finie (dans ce cas, les conditions (3) et (4) deviennent d'ailleurs identiques). Il existe encore un assez grand nombre de variantes de ces conditions, que le lecteur pourra énoncer et établir à titre d'exercice, en s'inspirant de **(4, §2, No 2)**.

THÉORÈME 1. *Soit L un espace de Banach. Pour que L satisfasse aux conditions équivalentes de la proposition* 1, *il faut et il suffit que L soit isomorphe (avec sa norme) à un espace $L^1(\mu)$ construit sur une mesure μ convenable (sur un espace localement compact convenable).*

La condition est suffisante, car un espace L^1 satisfait à la condition 1 de proposition 1, en vertu de l'identification

$$L^1 \hat{\otimes} E = L^1_E$$

(espace de fonctions vectorielles μ-intégrables, à valeurs dans E) pour tout espace de Banach E **(4, §2, No 1, Théorème 3)**, et du fait que l'inclusion

$$L^1_F \subset L^1_E$$

respecte la norme. Montrons la réciproque. Supposons donc prop. 1 vérifiée, L' est alors un espace (N), de plus, étant un dual, sa boule unité admet un point extrémal en vertu du théorème de Krein-Milman. Le théorème de Nachbin nous apprend que L' est isomorphe avec sa norme a un espace $C(K)$ (K compact convenable, d'ailleurs stonien, mais peu nous importera). Par cet isomorphisme, L devient un sous-espace vectoriel fermé du dual de $C(K)$, espace des mesures sur K. On va alors prouver le théorème suivant, qui a son intérêt propre:

THÉORÈME 2. *Soit K un espace localement compact, $C = C_0(K)$ l'espace des fonctions continues nulles à l'infini sur K, muni de la norme uniforme. Supposons que C soit isomorphe (avec sa norme) au dual d'un espace de Banach L; on peut donc supposer $L \subset C'$, l'isomorphisme de C avec L' étant donné par l'accouplement entre C et le sous-espace L de son dual. Sous ces conditions, K est un espace*

compact stonien, C est isomorphe à une algèbre de von Neumann[3], *et L est exactement l'ensemble des mesures* normales *sur K, i.e. des combinaisons linéaires de mesures positives μ qui ont la propriété suivante: Pour toute partie filtrante croissante A de C, de borne supérieure f dans C, on a* $\mu(f) = \lim \mu(f_\iota)$ (la limite étant prise suivant le filtre des sections croissantes dans A).

Il en résulte bien que L est isomorphe à un espace $L^1(\mu)$ construit sur une mesure μ convenable, comme il résulte par exemple d'un théorème classique de Kakutani **(7)**.

Soit C^+ le cône positif de C, je dis qu'il est faiblement fermé (par topologie faible, nous entendons $\sigma(C, L)$). En vertu d'un théorème classique de Banach-Dieudonné **(1**, No. 10, th. 5; nous utilisons le corollaire bien connu qu'une partie *convexe* du dual E' d'un espace $\{F\}$ est faiblement fermée si et seulement si son intersection avec toute partie faiblement compacte l'est**)**, il suffit de prouver qu'il en est ainsi de son intersection avec le boule unite de C, or cette dernière est l'ensemble des f telles que $\|f - g\| \leqslant 1$ pour toute g égale à 0 ou 1 d'où notre affirmation (puisque la boule unité de C est évidemment faiblement fermée, et même faiblement compacte). Il en résulte que pour toute $f \in C$, l'ensemble des éléments de C qui sont majorés (resp. minorés) par f est faiblement fermé. Soit, pour $0 \leqslant f \leqslant 1$, A_f l'ensemble des $g \in C$ telles que $f \leqslant g \leqslant 1$, A_f est aussi identique à l'intersection de la boule unité de C et de l'ensemble des éléments de C minorés par f, donc est faiblement fermé, donc faiblement compact puisque borné. Comme l'intersection d'un nombre fini des A_f est non vide, les A_f ont une intersection non vide. Donc la fonction 1 appartient à C, i.e., *K est compact.* Soit A un ensemble filtrant majoré dans C, soit A' l'ensemble des majorants de A, et pour $f \in A, f' \in A'$ soit $A_{f,f'}$ l'ensemble des $g \in C$ comprises entre f et f'. Les $A_{f,f'}$ constituent une base de filtre formés d'ensembles faiblement compacts, donc ont une intersection I non vide. Un élément ϕ de I majore A, donc appartient à A', et minore A', donc est un plus petit élément de A', donc c'est une borne supérieure de A dans C. Cela montre que *K est stonien.* D'autre part, I étant réduit à un point, on voit que la base de filtre formé des ensembles faiblement compacts $A_{f,f'}$ converge faiblement vers ϕ, à fortiori le filtre des sections croissantes de A (qui est plus fin) converge faiblement vers ϕ. Cela prouve en particulier que *tout élément positif de L est une forme normale.* Prouvons maintenant que pour tout $f \in C$, l'opération de multiplication par f dans C est faiblement continue. K étant stonien, les combinaisons linéaires des fonctions caractéristiques d'ensembles à la fois ouverts et fermés (i.e. les idempotants de l'algèbre C) sont denses dans C, ce qui ramène aussitôt au cas où f est une telle fonction caractéristique (car les endomorphismes faiblement continus de C forment un sous-espace fermé de l'espace normé $L(C)$ de tous

[3]On appelle *algèbre de von Neumann* une algèbre autoadjointe d'opérateurs continus dans un espace de Hilbert, avec unité et faiblement fermée. Les scalaires peuvent être réels ou complexes.

les endomorphismes continus). Alors l'application correspondante de C est un projecteur dont l'image est fC et le noyau $f'C$ (ou $f' = 1 - f$), et il est bien connu[4] qu'il suffit de prouver que ces espaces sont faiblement fermés. Il suffit de le voir pour le premier (le second est du même type), et en vertu du théorème de Banach-Dieudonné déjà utilisé, il suffit de montrer que son intersection avec la boule unité est faiblement fermée. Or cette intersection est aussi l'ensemble des g telles que $-f \leqslant g \leqslant f$, et est par suite faiblement fermé en vertu de ce qui a été dit plus haute. Soit $f \in C$. Dire que $f \to fg$ est faiblement continue, signifie aussi que pour tout $\mu \in L$, la forme $g \to \langle fg, \mu \rangle$ sur C est faiblement continue, i.e. appartient à L. Cette forme n'est autre que la mesure $f\mu$ de densité f par rapport à μ. Donc L est stable par multiplication par les $f \in C$. Or, il est bien connu (et facile à vérifier) que pour toute mesure μ sur K, $|\mu|$ est limite au sens de la norme de mesures $f\mu$, $f \in C$. Comme L est un sous-espace fortement fermé de C', on en conclut que $\mu \in L$ implique $|\mu| \in L$, donc que L est un sous-lattice de C'. En particulier, L est engendré par ses éléments positifs, donc *les $\mu \in L$ sont des formes normales*. Pour tout élément positif μ de L, considérons l'espace $L^2(\mu)$, soit H leur produit hilbertien (quand μ parcourt L^+); pour $f \in C$, soit U_f l'opérateur dans H qui sur chaque facteur $L^2(\mu)$ se réduit à la multiplication par f. On a ainsi une représentation unitaire de C dans H. Cette représentation est biunivoque, car si U_f est nul, on voit aussitôt que f est orthogonale à toutes les $\mu \in L^+$, donc orthogonale à L, donc nulle (puisque C s'identifie au dual de L). De plus, pour $x \in H$, $y \in H$, la forme $\langle U_f x, y \rangle$ sur C est dans L. En effet, L étant fortement fermée dans le dual de C, il suffit de le prouver quand x et y parcourent une partie totale de H, en particulier quand x et y appartiennent à un des espaces $L^2(\mu)$. On peut supposer que c'est le même pour x et y (autrement on a une forme nulle sur C). Alors on a

$$\langle U_f x, y \rangle = \int fgh\,d\mu$$

où g et h sont dans $L^2(\mu)$, i.e. en introduisant $\phi = gh \in L^1(\mu)$, on a $\langle U_f x, y \rangle = \langle f, \phi\mu \rangle$. Or les $\phi\mu$, $\phi \in L^1(\mu)$ sont dans L, car il en est ainsi, on l'a vu, des $\phi\mu$ avec $\phi \in C$, or l'image de C dans $L^1(\mu)$ est dense, et L est fermé dans C'. On a prouvé que la représentation U_f est continue pour la topologie faible de C et la topologie "faible" de l'espace $L(H)$ des endomorphismes de H.

[4] De façon générale, soit E un espace (F), soient F et G deux sous-espaces vectoriels faiblement fermés de E', dont la somme est directe et égale à E'. Alors E'_i faible est somme directe topologique de ses sous-espaces vectoriels F et G. En effet, F et G sont resp. les polaires des sous-espaces vectoriels F° et G° de E, soit u l'application linéaire de E dans $(E/F^\circ) \times (E/G^\circ)$ dont les deux composantes sont les applications canoniques de E dans les quotients. Le dual du produit est le produit des duals, i.e. $F \times G$, et la transposée de u est l'application naturelle $(y, z) \leqslant y + z$ de $F \times G$ dans E'. Elle est biunivoque et sur par hypothèse, et on sait qu'il en résulte (toujours une conséquence du théorème de Banach-Dieudonné!) que u est un isomorphisme sur. Cela signifie aussi que F° et G° sont supplémentaires topologiques, d'ou enfin la conclusion voulue: leurs polaires F et G sont supplémentaires topologiques dans E' faible.

Par suite elle transforme la boule unité de C en une partie faiblement compacte de $L(H)$. Il en résulte (**2**, th. 8[5]) que l'image de C dans $L(H)$ est faiblement fermée, i.e. est une *algèbre de von Neumann*. Ainsi C est isomorphe à une algèbre de von Neumann. On sait que si C est une C^*-algèbre isomorphe à une algèbre de von Neumann, et si on désigne par C_* l'espace des formes normales sur C, alors C s'identifie avec sa norme au dual de C_*. Or on a vu que $L \subset C_*$. Cela implique $L = C_*$, car si on avait $L \neq C_*$. Il existerait une forme linéaire sur C_* non identiquement nulle, mais nulle sur le sous-espace L (Hahn-Banach), i.e. un élément de C non nul, qui s'annule sur L; absurde puisque on suppose que C s'identifie aussi au dual de L. Le théorème 2 est complètement démontré.

Remarque 2. Si on n'avait eu en vue que le théorème 1, une partie de la démonstration du théorème 2 aurait suffi, puisqu'on savait déjà (par le théorème de Nachbin) qu'on pouvait supposer K compact stonien (ce qui dispense de le démontrer). Il suffisait alors de prouver seulement que les multiplications dans C sont des opérations faiblement continues, d'où résulte que L est stable sous les multiplications par les $f \in C$, donc est un "idéal" au sens latticiel, auquel le théorème de Kakutani s'applique alors (et même un cas particulier assez facile de ce théorème).

Remarque 3. On peut se demander si le théorème 2 se généralise à toute C^*-algèbre (non nécessairement abélienne comme dans notre énoncé): Si une telle algèbre C est isomorphe (avec sa norme) au dual d'un sous-espace L de C', est-il vrai que C est isomorphe à une algèbre de von Neumann, et que L est exactement l'ensemble des formes normales sur C ? Si on suppose L stable sous l'involution naturelle de C' (elle le sera peut-être automatiquement), alors nos raisonnements précédents prouvent encore que C a une unité, que toute partie filtrante croissante majorée de la partie hermitienne de C a une borne supérieure, et que le filtre des sections croissantes converge vers cette dernière pour la topologie faible définie par L. La seule difficulté est dans la question si L est engendré par sa partie positive, le raisonnement donné dans le cas commutatif ne vaut pas tel quel. Il semble probable cependant que la technique des C^*-algèbres jointe au théorème de Banach-Dieudonné doive permettre de donner une réponse affirmative à notre question.

Remarque 4. Le théorème 2 montre qu'un espace $C_s(K)$ ne peut être essentiellement que "d'une seule manière" le dual d'un espace de Banach L (de façon précise, si on identifie L à un sous-espace de C', L est uniquement déterminé). Il n'en est pas ainsi dans le cas général. Considérons par exemple l'espace $E = l^1$, je dis qu'il existe une infinité de sous-espaces vectoriels fermés F dans $E' = l^\infty$ tels que E s'identifie (avec sa norme) au dual de F. En effet, il est immédiat qu'il existe une infinité de topologies T sur l'ensemble N des entiers qui fassent de N un espace localement compact N_T. Pour toute telle

[5]Ce théorème est encore une conséquence du théorème de Banach-Dieudonné.

topologie, $C_0(N_T)$ est un sous-espace vectoriel fermé de l^∞, dont le dual est l'espace des mesures bornées sur l'espace localement compact dénombrable N_T. Or il est bien connu que toute mesure sur un espace N_T est discrète, donc que le dual en question s'identifie aussi à $l^1(N_T) = l^1$.

On peut se demander si dans l'énoncé de la condition 4 de proposition 1, on ne peut pas remplacer E'' par E lui-même. La proposition qui suit donne une réponse complète:

PROPOSITION 2. *Soit L un espace de Banach qui jouit des propriétés de la proposition* 1, *i.e.* (*théorème* 1) *est isomorphe* (*avec sa norme*) *à un espace $L^1(\mu)$ construit sur une mesure μ convenable sur un espace localement compact convenable. Pour que toute application linéaire continue u de L dans un quotient E/F d'un espace de Banach quelconque par un sous-espace vectoriel fermé F, se relève en une application linéaire continue v de L dans E, il faut et il suffit que μ soit discrète, ou encore que L soit isomorphe* (*avec sa norme*) *à un espace $l^1(I)$* (*espace des suites sommables de scalaires sur un ensemble d'indices I convenable*). *S'il en est ainsi, alors on peut trouver même un relèvement v de norme arbitrairement voisine de celle de u.*[6]

Démonstration. Si μ est discrète, i.e. somme d'une famille localement sommable de masses ponctuelles, il est bien connu (et facile à voir) que $L^1(\mu)$ est isomorphe avec sa norme à un espace $l^1(I)$ (on prendre pour I l'ensemble d'indices de la famille précédente). D'autre part, pour un espace de Banach quelconque G, on sait que l'espace $L(l^1(I), G)$ des applications linéaires continues de $l^1(I)$ dans G s'identifie, avec sa norme, à l'espace $l^\infty(I, G)$ des familles bornées $(x_i)_{i \in I}$ d'éléments de G sur l'ensemble d'indices I, muni de la norme uniforme

$$\sup_{i \in I} ||x_i||;$$

à (x_i) correspondant l'application linéaire

$$(\lambda_i) \to \sum \lambda_i x_i$$

de $l^1(I)$ dans G. Soit alors u une application linéaire continue de $l^1(I)$ dans E/F, donc définie par une famille d'éléments (x_i) de E/F de norme $\leqslant ||u||$, et soit $\epsilon > 0$. Pour tout $i \in I$, soit y_i un représentant de x_i dans E de norme

$$||y_i|| \leqslant (1 + \epsilon)||x_i|| \leqslant (1 + \epsilon)||u||.$$

Alors (y_i) définit une application linéaire v de norme $\leqslant (1 + \epsilon)||u||$ de $l^1(I)$ dans E, et il est immédiat que v relève l'application donnée u. Cela prouve que quand L est du type $L^1(\mu)$ avec une mesure μ discrète, il possède la propriété envisagée dans la proposition 2. (Ce résultat, ainsi que la démonstration donnée ici, sont dus àG. Köthe). Réciproquement, supposons $L = L^1(\mu)$

[6]Bien entendu, même dans le cas où L est réduit au corps des scalairès, on ne pourra en général trouver un relèvement v de *même* norme que u, car en général un élément x du quotient E/F n'a pas de représentant dans E de même norme que x.

tel que toute application linéaire continue de L dans un quotient E/F puisse se relever en une application linéaire continue de L dans E. On sait que tout espace de Banach, en particulier L, est isomorphe à un espace quotient E/F d'un espace $E = l^1(I)$ construit sur un ensemble d'indices I convenable (par exemple I = la boule unité de L). Appliquant l'hypothèse sur L, on voit qu'il existe une application linéaire continue v de L dans E, inverse à droite de l'homomorphisme donné de E sur L, donc L est isomorphe (du point de vue vectoriel-topologique) à un facteur direct de E, donc à un sous-espace de $E = l^1(I)$. Or, dans $l^1(I)$ les parties faiblement compactes sont compactes, comme il est bien connu; il en est donc de même dans tout sous-espace vectoriel topologique de $l^1(I)$ et en particulier de $L = L^1(\mu)$. Mais on sait que ceci exige que la mesure μ soit discrète (voir par exemple dernier alinéa de **(5)**). Ceci achève la démonstration de proposition 2.

Remarque 5. On peut considérer aussi les espaces de Banach L qui satisfont aux variantes vectorielles-topologiques des condition (de nature métrique) de la proposition 1, telle la variante suivante de la condition 4: Toute application linèaire continue de L dans un quotient E/F d'espace de Banach se relève en une application linéaire continue de L dans E''. Utilisant le fait que l'espace de Banach L est isomorphe à un espace quotient d'un espace $l^1(I)$, on trouve facilement, en procédant comme dans la démonstration précédente, que L satisfait à la condition énoncée si et seulement si son bidual est isomorphe à un facteur direct d'un espace $L^1(\mu)$ (de facon précise, il sera isomorphe à un facteur direct du bidual de $l^1(I)$, lequel est lui-même isomorphe à un espace $L^1(\mu)$, comme il résulte par exemple du théorême de Kakutani). J'ignore si un tel espace est même isomorphe, du point de vue vectoriel-topologique, à un espace L^1, tout comme on ignore si un espace de Banach C qui satisfait à la variante vectorielle-topologique de la propriété de Nachbin (qui est de nature métrique), est déjà isomorphe (du point de vu vectoriel-topologique) à un espace (N).

Remarque 6. Il n'y a guère de doute que le théorème de Nachbin, ainsi que les théorèmes de ce travail, sont valables tels quels dans le cas de scalaires *complexes* au lieu de scalaires réels. Il n'est pas certain cependant que cette extension soit immédiate.

Un problème. Soit C un espace de Banach, considérons la condition suivante: 1'. Quel que soit l'espace de Banach E et le sous-espace vectoriel fermé F, l'application canonique

$$C \check{\otimes} E \to C \check{\otimes} E/F$$

est un homomorphisme métrique du premier sur le second. (Je suis les notations de **(4)**, où j'emploie le signe $\vee$ à la place du signe $\hat{\otimes}$ utilisé dans mon travail **(6)**. On voit sans difficulté qu'une condition équivalente est: 2'. l'espace C' dual de C satisafit aux conditions de proposition 1, i.e. (théorème 1)

est isomorphe à un espace L^1. Autres conditions équivalentes: Toute application linéaire continue (resp. compacte) d'un sous-espace normé F d'un espace de Banach quelconque E, se prolonge en une application linéaire continue de norme égale de E dans le bidual C'' (resp. se prolonge en une application linéaire compacte de norme arbitrairement voisine de E dans C) **(4, §2, N° 2)**. Nous dirons qu'un espace qui satisfait aux propriétés précédentes est un *espace de classe C*, tandis qu'un espace qui satisfait aux conditions de proposition 1, i.e. isomorphe à un espace L^1, est appelé un *espace de classe L*. Ainsi E est de classe C (resp. L) si et seulement si son dual est de classe L (resp. C). Par exemple l'espace $C(K)$ des fonctions continues sur un compact, avec la norme uniforme, plus généralement l'espace $C_0(M)$ des fonctions continues nulles à l'infini sur un espace localement compact M, sont de classe C (car on voit en vertu du théorème de Kakutani que le dual est de classe L). Plus généralement, les "abstract M-spaces" de Kakutani sont des espaces C, car Kakutani montre que leurs duals sont des espaces L **(8, p. 1021, théorème 15)**. Il existe d'ailleurs des espaces C qui ne rentrent pas dans ce type, p.ex. l'espace des fonctions continues *symétriques* sur la circonférence unité du plan complexe (qui n'est pas un espace (μ), car les points extrémaux de la boule unité du dual forment un espace compact *connexe*, canoniquement isomorphe au quotient de la circonférence unité par identification des points diamétralement opposés.) Ces exemples suggèrent la validité de l'énoncé suivant: Les espaces de classe C sont exactement ceux qui sont isomorphes (avec leur norme) à un sous-espace E d'un espace $C(K)$ (K espace compact) formé par les fonctions qui satisfont à des conditions de la forme: $f(x_t) = \lambda_t f(y_t)$ (x_t et y_t éléments de K, λ_t scalaires, qu'on peut supposer compris entre -1 et $+1$). (Bien entendu, si l'espace E de classe C est donné, on prendra pour K l'adhérence faible de l'ensemble des points extrémaux de la boule unité de E', et on est ramené à un problème d'approximation.)

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Ajouté pendant la correction des épreuves. Monsieur Lowdenslager m'a fait observer les faits suivants. La généralisation du théorème 2 conjecturée dans la remarque 3 a été prouvée récemment par *Sakai*, dans un papier qui sera publié dans le Pacific Journal of Mathematics. Le théorème de Nachbin cité au début de ce travail a été prouvé indépendamment par D. B. Goodner; la plus jolie preuve connue semble être celle de Kelley (Banach spaces with the extension property, Trans. Amer. Math. Soc., *72* (1952), 323–326), qui ne suppose pas que la boule unité admette un point extrémal.

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SUR LA FORME HERMITIENNE CANONIQUE DES ESPACES HOMOGENES COMPLEXES

J. L. KOSZUL

Introduction. Soit G un groupe de Lie connexe et soit B un sous-groupe fermé de G. Supposons que la variété homogène G/B ou G opère transitivement à gauche soit munie d'une structure complexe invariante par les opérations de G, et qu'il existe sur G/B un volume invariant par G. Ce volume étant unique à un facteur constant près, la forme hermitienne de G/B qui est canoniquement associée au volume et à la structure complexe se trouve entièrement déterminée par la structure complexe de G/B et le groupe transitif G **(5)**. C'est cette forme hermitienne que nous appelons la *forme hermitienne canonique* de l'espace homogène complexe G/B. L'intérêt que présente cette forme pour la détermination des domaines bornés homogènes a été souligné par E. Cartan: une condition nécessaire pour que G/B soit un domaine borné est en effet que cette forme soit définie positive **(1)**.

Notre premier but est ici de calculer cette forme canonique à partir de données infinitésimales qui sont: l'algèbre de Lie $\mathfrak{g}$ de G, la sous-algèbre de $\mathfrak{g}$ qui correspond à B et un endomorphisme de $\mathfrak{g}$ définissant la structure complexe invariante de G/B. Les paragraphes 1, 2, et 3 contiennent des préliminaires à ce calcul qui est achevé au §4. La formule obtenue met en évidence le rôle joué par une certaine forme invariante de degré 1 sur G qui est definie par la structure complexe de G/B. Au §5, on en déduit des conditions nécessaires pour que la forme hermitienne de G/B soit non dégénérée. Une étude détaillée du cas où G est semi-simple et B compact fait l'objet du §6. Elle montre que des relations étroites existent dans ce cas entre la compacité de G/B et les carrés négatifs de la forme canonique. Les résultats obtenus prouvent que les domaines bornés homogènes dont le groupe des automorphismes est semi-simple sont des domaines bornés symétriques au sens de E. Cartan **(3)**. Les problèmes que nous envisageons ici du point de vue de la "Théorie des groupes de Lie" se rattachent à un grand nombre de résultats récents de Géométrie différentielle **(2; 12)**.

1. Champs de vecteurs projetables. Soit V une variété différentiable de classe ∞ et soit F l'algèbre sur le corps des nombres réels des fonctions à valeurs réelles indéfiniment différentiables sur V. On notera C le F-module des champs de vecteurs[1] sur V et $T(C)$ l'algèbre tensorielle du F-module C. Les éléments de degré p de $T(C)$ constituent le F-module T_0^p des champs de

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[1]Dans tout ce mémoire, on ne considère que des variétés et des champs de tenseurs indéfiniment différentiables.

tenseurs de type $(0, p)$ sur V. Le F-module $T_p^q = \mathrm{Hom}_F(T_0^p, T_0^q)$ est le module des champs de tenseurs de type (p, q) sur V. Il y aura parfois avantage à considérer un élément $t \in T_p^q$ comme une fonction F-linéaire de p champs de vecteurs X_j $(j = 1, 2, \ldots, p)$ à valeurs dans T_0^q; on écrira alors cette fonction $t(X_1, \ldots, X_p)$. Par abus de langage, les champs de tenseurs sur V seront appelés des *tenseurs de V*.

Pour tout champ de vecteurs X, on notera $\theta(X)$ la transformation infinitésimale associée à X. Elle transforme un tenseur t de type (p, q) en un tenseur $\theta(X).t$ de même type. On a

$$(1,1)\quad (\theta(X) \,.\, t)(X_1, \ldots, X_p) = \theta(X) \,.\, (t(X_1, \ldots, X_p)) - \sum_{j=1}^{p} t(X_1, \ldots, [X, X_j], \ldots, X_p)$$

quels que soient les champs $X_j \in C$ $(j = 1, 2, \ldots, p)$.

Soit maintenant π une application différentiable de V *sur* une variété différentiable W. A tout tenseur t de type $(p, 0)$ sur W, l'application π fait correspondre un tenseur de même type sur V noté $\pi^*.t$. L'application π^* est un isomorphisme. Soit $F' \subset F$ l'image par π^* l'algèbre des fonctions différentiables de W. Un champ de vecteurs X sur V sera dit *projetable* si $\theta(X)$. $F' \subset F'$. Si X est projetable, il existe sur W un champ de vecteurs X' et un seul tel que $\theta(X)\, \pi^*.f = \pi^*\theta(X').f$ pour toute fonction différentiable f sur W. Ce champ X' sera appelé la *projection* de X et sera noté $\pi.X$. L'application π vérifie les propriétés suivantes:

(1.2) *si X et Y sont projetables, alors $X + Y$ et $[X, Y]$ sont projetables et* $\pi.(X + Y) = \pi.X + \pi.Y, \pi.[X, Y] = [\pi.X, \pi.Y]$,

(1.3) *si X est projetable et si f est une fonction différentiable sur W, alors* $(\pi^*.f)\ X$ *est projetable et* $\pi.((\pi^*.f)\ X) = f(\pi.X)$,

(1.4) *si t est un tenseur de type $(p, 0)$ sur W et si les X_j $(j = 1, 2, \ldots, p)$ sont des champs de vecteurs projetables sur V, on a*

$$(\pi^*.t)(X_1, \ldots, X_p) = \pi^*.(t(\pi.X_1, \ldots, \pi.X_p)),$$

(1.5) *pour tout tenseur t de type $(p, 0)$ de W et tout champ de vecteurs projetable X sur V,* $\theta(X)\, \pi^*.t = \pi^*\theta(\pi.X).t$.

Si V est un espace fibré différentiable *localement trivial* et si $\pi: V \to W$ est la projection de V sur sa base, alors tout champ de vecteurs sur W est projection d'un champ de vecteurs sur V.

Par la suite, on utilisera ces champs projetables dans le cas où V est un groupe de Lie connexe G et W un espace homogène de groupe G. Plus précisément, un sous-groupe fermé B sera donné[2] dans G et $W = G/B$ sera la variété quotient de G par la relation d'équivalence: $s \backsim s'$ lorsque $s'\, s^{-1} \in B$. La projection $\pi: G \to G/B$ associe à chaque $s \in G$ sa classe d'équivalence consid-

[2]Le sous-groupe B ne sera pas supposé connexe.

érée comme élément de G/B. On désignera par $\mathfrak{g}$ l'algèbre de Lie des champs de vecteurs invariants à *gauche* sur G et par $\mathfrak{b}$ la sous-algèbre des $X \in \mathfrak{g}$ qui, au point neutre e de G, sont tangents à B. Cette sous-algèbre $\mathfrak{b}$ est canoniquement isomorphe à l'algèbre de Lie des champs de vecteurs invariants à gauche sur B. La translation à droite par un élément $s \in G$ transforme tout champ de vecteurs X sur G en un champ noté Xs. Pour qu'un champ de vecteurs X sur G soit projetable sur G/B, il faut et il suffit que, pour tout $s \in B$, le champ de vecteurs $Xs - X$ soit dans le F-module $F\mathfrak{b}$ engendré par les éléments de $\mathfrak{b}$. Les champs de vecteurs dont la projection est nulle sont les éléments de $F\mathfrak{b}$. En particulier, les champs de vecteurs invariants à *droite* sur G sont projetables. Les transformations infinitésimales associées à leurs projections correspondent aux opérations de G opérant à gauche sur G/B.

Soit $T(e')$ l'espace vectoriel des vecteurs tangents à G/B au point $e' = \pi.e$. Les opérations de G sur G/B définissent une représentation linéaire de B dans $T(e')$ qui sera notée à gauche. Les $X \in \mathfrak{g}$ ne sont pas projetables en général, mais en associant à tout $X \in \mathfrak{g}$ l'image par l'application linéaire tangente à π du vecteur X_e défini par X au point e, on obtient une application linéaire π' de $\mathfrak{g}$ sur $T(e')$ dont le noyau est $\mathfrak{b}$. Pour tout $s \in B$, on a alors $s(\pi'.X) = \pi'.(X\,s^{-1})$ quel que soit $X \in \mathfrak{g}$.

2. Structure complexe invariante sur un espace homogène. Les notations restant celles du paragraphe précédent, supposons que la variété G/B soit munie d'une structure presque complexe invariante par les opérations de G. Cette structure est définie par un tenseur I de type $(1, 1)$ sur G/B, c'est à dire un endomorphisme du module des champs de vecteurs sur G/B, qui est invariant par G et vérifie la condition $I^2.U = -U$ pour tout champ de vecteurs U. Soit $I_{e'}$ l'endomorphisme de $T(e')$ défini par I au point e'. De l'invariance de I par G résulte que $I_{e'}$ commute avec les opérations de B dans $T(e')$. Soit J un endomorphisme de l'espace $\mathfrak{g}$ tel que

$$\pi' J = I_{e'} \pi', \tag{2.1}$$

$$J.\mathfrak{b} = (0). \tag{2.2}$$

Des propriétés de $I_{e'}$ résultent que

$$J^2.X = -X \pmod{\mathfrak{b}} \text{ pour tout } X \in \mathfrak{g} \tag{2.3}$$

$$(J.X)s = J.(Xs) \pmod{\mathfrak{b}} \text{ pour tout } X \in \mathfrak{g} \text{ et } s \in B. \tag{2.4}$$

Réciproquement, soit J un endomorphisme de $\mathfrak{g}$ vérifiant les conditions (2.2), (2.3) et (2.4). Il existe un endomorphisme $I_{e'}$ de $T(e')$ et un seul qui vérifie la condition (2.1). Cet endomorphisme est de carré -1 et commute avec les opérations de B. Il définit donc un tenseur I de type $(1, 1)$ sur G/B invariant par G et tel que $I^2.U = -U$ pour tout champ de vecteurs U sur G/B, c'est à dire une structure presque complexe sur G/B.

Le F-module des champs de vecteurs sur G étant un module libre qui admet pour base une base de l'espace $\mathfrak{g}$, l'endomorphisme J de $\mathfrak{g}$ est la restric-

tion d'un tenseur de type (1, 1) sur G qui est invariant par les translations à gauche et que l'on notera encore J. D'après (2.4), $J.(Xs) - (J.X)s \in F\mathfrak{b}$ pour tout $s \in G$ et comme d'après (2.2) $J.(F\mathfrak{b}) = (0)$, on voit que *si X est un champ de vecteurs projetable sur G, alors $J.X$ est également projetable.* De plus, J et I étant invariant par G, la condition (2.1) entraîne que l'on a alors

$$\pi.(J.X) = I.(\pi.X). \tag{2.5}$$

Supposons maintenant que la structure presque complexe donnée sur G/B soit une structure complexe. Le tenseur R de type (2, 1) sur G/B défini par

$$R(U, V) = [U, V] + I.[I.U, V] + I.[U, I.V] - [I.U, I.V]$$

où U et V sont des champs de vecteurs sur G/B est alors **(8)** égal à 0. Associons comme précédemment à I un tenseur J sur G et posons

$$S(X, Y) = [X, Y] + J.[J.X, Y] + J.[X, JY] - [J.X, J.Y]$$

quels que soient les champs de vecteurs X, Y sur G. Si X et Y sont des champs invariants à droite sur G, donc projetables, alors $S(X, Y)$ est projetable et $\pi.S(X, Y) = R(\pi.X, \pi.Y) = 0$ d'après (1.2) et (2.5). Par conséquent, le champ de vecteurs $S(X, Y)$ est dans $F\mathfrak{b}$. Nous allons voir que ceci est encore vérifié pour des champs de vecteurs X, Y arbitraires sur G. Il suffit pour cela d'observer que, pour toute fonction $f \in F$, on a

$$S(X, fY) = fS(X, Y) + (\theta(x).f)(Y + J^2.Y)$$

quels que soient les champs[3] de vecteurs X, Y. D'après (2.3), $Y + J^2.Y \in F\mathfrak{b}$, par suite $S(X, fY) = fS(X, Y) \bmod F\mathfrak{b}$. Comme $S(X, Y) = -S(Y, X)$ et que le F-module des champs de vecteurs sur G est engendré par les champs invariants à droite, ceci prouve que la relation $S(X, Y) \in F\mathfrak{b}$ est vérifiée quels que soient X et Y. En particulier, si X, $Y \in \mathfrak{g}$, $S(X, Y) \in F\mathfrak{b} \cap \mathfrak{b} = \mathfrak{b}$. On voit donc que, si I définit sur G/B une structure complexe, alors l'endomorphisme J de $\mathfrak{g}$ vérifie la condition

$$[X, Y] + J.[J.X, Y] + J.[X, J.Y] - [J.X, J.Y] \in \mathfrak{b} \tag{2.6}$$

quels que soient[4] X, $Y \in \mathfrak{g}$.

Réciproquement, si l'on part d'un endomorphisme J de $\mathfrak{g}$ qui vérifie les conditions (2.2), (2.3), (2.4) et (2.6), la structure presque complexe sur G/B qui lui correspond est une structure complexe invariante par G. En effet, $S(X, Y) \in F\mathfrak{b}$ quels que soient les champs de vecteurs X, Y sur G et par suite, si X et Y sont projetables, $R(\pi.X, \pi.Y) = 0$. Tout champ de vecteurs sur

[3] D'après la formule $[X, fY] = f[X, Y] + (\theta(X).f)Y$, valable quels que soient les champs de vecteurs X, Y et la fonction f.

[4] Des conditions (2.2) et (2.3) résulte que $J(J^2 + 1) = 0$. On vérifie facilement que (2.6) équivaut à la condition suivante: si J^C est le prolongement de J à l'algèbre $\mathfrak{g}^C$ complexifiée de $\mathfrak{g}$, le sous-espace des zéros de $J^C(J^C - i)$ est une sous-algèbre de $\mathfrak{g}^C$.

G/B étant projection d'un champ de G, on a $R = 0$ c'est à dire que le tenseur I vérifie la condition d'intégrabilité. Comme I est invariant par G, il est analytique réel pour la structure analytique réelle canonique de G/B. On sait que dans ces conditions, I définit sur G/B une structure *complexe* **(8)**.

3. Une expression de la forme hermitienne associée à un volume. Nous supposerons ici que V est une variété analytique complexe de dimension réelle $2n$ sur laquelle est donnée une forme différentielle réelle ω de degré $2n$ partout de rang $2n$ sur V. Un champ de vecteurs complexes sur V est une combinaison $X + (-i)^{\frac{1}{2}}Y$ où X et Y sont des champs de vecteurs réels. Pour tout champ de vecteurs complexes $\mathfrak{X}$ sur V, on notera $\partial.\mathfrak{X}$ la *divergence* de $\mathfrak{X}$, c'est à dire la fonction différentiable à valeurs complexes définie par $\theta(\mathfrak{X}).\omega = (\partial.\mathfrak{X})\omega$. On vérifie facilement les formules

$$\theta(\mathfrak{X})\,\partial.\mathfrak{Y} - \theta(\mathfrak{Y})\,\partial.\mathfrak{X} = \partial.[\mathfrak{X}, \mathfrak{Y}], \qquad \partial.(f\mathfrak{X}) = f(\partial.\mathfrak{X}) + \theta(\mathfrak{X}).f, \tag{3.1}$$

qui sont valables quels que soient les champs $\mathfrak{X}$, $\mathfrak{Y}$ et quelle que soit la fonction complexe f.

Soient z_j $(j = 1, 2, \ldots, n)$ des coordonnées locales complexes au voisinage d'un point de V et soient $\mathfrak{Z}_i$ les champs de vecteurs complexes tels que $\theta(\mathfrak{Z}_j).f = \partial f/\partial z_j$ pour toute fonction différentiable à valeurs complexes définie au voisinage de ce point. Le volume ω s'écrit localement

$$\omega = K\, dz_1 \wedge \ldots dz_n \wedge d\bar{z}_1 \wedge \ldots d\bar{z}_n$$

où K est une fonction différentiable à valeurs réelles ou imaginaires pures suivant la parité de n. On a, pour tout i,

$$\partial\,.\,\mathfrak{Z}_j = \frac{\partial \operatorname{Log} K}{\partial z_j}, \qquad \partial\,.\,\bar{\mathfrak{Z}}_j = \frac{\partial \operatorname{Log} K}{\partial \bar{z}_j}.$$

Soit $\mathfrak{Y}$ un champ de vecteurs analytique sur V qui s'écrira localement

$$\mathfrak{Y} = \sum_{j=1}^{n} b_j \mathfrak{Z}_j$$

où les b_i sont des fonctions analytiques et soit $\mathfrak{X}$ un champ de vecteurs de type analytique sur V, c'est à dire s'écrivant localement $\mathfrak{X} = \sum a_j \mathfrak{Z}_j$ où les a_j sont des fonctions différentiables à valeurs complexes. D'après (3.1), on a

$$\partial\,.\,\bar{\mathfrak{Y}} = \sum_{j=1}^{n} \frac{\partial \bar{b}_j}{\partial \bar{z}_j} + \sum_{k=1}^{n} \frac{\partial \operatorname{Log} K}{\partial \bar{z}_k}\, \bar{b}_k.$$

Puisque $\partial \bar{b}_j/\partial z_k = 0$ quels que soient j et k, on a donc

$$\theta(\mathfrak{X})\,.\,(\partial\,.\,\bar{\mathfrak{Y}}) = \sum_{j,k=1}^{n} \frac{\partial^2 \operatorname{Log} K}{\partial z_j \partial \bar{z}_k}\, a_j \bar{b}_k.$$

Or le tenseur dont les composantes sont localement

$$\frac{\partial^2 \operatorname{Log} K}{\partial z_j \partial \bar{z}_k}$$

est la forme hermitienne **(1)** associé au volume ω. On voit donc que (3.2) *Si $\mathfrak{Y}$ est un champ analytique et si X est un champ de type analytique, alors $\theta(\mathfrak{X}).(\partial.\overline{\mathfrak{Y}}) = h(\mathfrak{X}, \overline{\mathfrak{Y}})$ où h désigne la forme hermitienne associée au volume ω.*

On observera que cette expression de h, dont le caractère intrinsèque nous sera commode dans la suite, cesse d'être valable pour des champs $\mathfrak{X}$, $\mathfrak{Y}$ quelconques.

La restriction de h aux champs de vecteurs réels est une forme bilinéaire symétrique réelle qui vérifie la condition $h(I.X, Y) + h(X, I.Y) = 0$, où I est le tenseur de type (1, 1) sur V qui définit la structure complexe. On désignera par a la forme antisymétrique réelle $a(X, Y) = h(X, IY)$. Si X et Y sont deux champs de vecteurs réels sur V et si $\mathfrak{X} = X - i\, I.X$ et $\mathfrak{Y} = Y - i\, I.Y$ sont les champs de type analytique qui leur correspondent, on a $4\, a(X, Y) = i\, h(\mathfrak{X} + \overline{\mathfrak{X}}, \mathfrak{Y} - \overline{\mathfrak{Y}}) = i\, (h(\overline{\mathfrak{X}}, \mathfrak{Y}) - h(\mathfrak{X}, \overline{\mathfrak{Y}}))$. Si Y est *conforme*, c'est à dire si $\mathfrak{Y}$ est analytique, la formule (3.2) montre donc que

$$2a(X, Y) = \theta(X).(\partial I.Y) - \theta(I.X).(\partial.Y).$$

Compte tenu de (3.1), il en résulte que:

(3.3) *Si X et Y sont deux champs de vecteurs conformes tels que $\partial.X = 0$ et $\partial.Y = 0$, alors $2a(X, Y) = \partial.\,(I\,[X, Y])$.*

4. Calcul de la forme hermitienne canonique des espaces homogènes complexes. Les notations étant celles du §1, supposons qu'il existe sur G/B un volume ω invariant par G. Une condition nécessaire et suffisante pour qu'il en soit ainsi est que la représentation linéaire de B dans $T(e')$ se fasse par des endomorphismes de déterminant 1. Le volume invariant, s'il existe, est déterminé à un facteur constant près. Etant donné un endomorphisme γ de l'espace $\mathfrak{g}$ tel que $\gamma.\mathfrak{b} \subset \mathfrak{b}$, on notera $\mathrm{Tr}_{\mathfrak{g}/\mathfrak{b}}\,\gamma$ la trace de l'endomorphisme de $\mathfrak{g}/\mathfrak{b}$ déduit de γ par passage au quotient. L'existence d'un volume invariant sur G/B implique $\mathrm{Tr}_{\mathfrak{g}/\mathfrak{b}}\,\mathrm{ad}(X) = 0$ pour tout[5] $X \in \mathfrak{b}$.

Soit X_j $(j = 1, 2, \ldots m)$ une base de l'espace $\mathfrak{g}$ telle que les X_j dont l'indice vérifie $2n < j \leqslant m$ constituent une base de $\mathfrak{b}$, les entiers m et $2n$ étant respectivement les dimensions de G et G/B. Soient ξ_j $(j = 1, 2, \ldots m)$ les formes de degré 1 invariantes à gauche sur G définies par $\xi_j(X_k) = \delta_{jk}$ (indice de Kronecker). L'image inverse $\Omega = \pi^*.\omega$ de la forme ω est égale, à un facteur constant près, à $\xi_1 \wedge \xi_2 \wedge \ldots \xi_{2n}$. On pourra supposer la base X_j choisie de sorte que ce facteur soit égal à 1.

LEMME 1. *Pour tout champ de vecteurs projetable X sur G, on a*

$$\pi^*\partial_\pi\,.\,X = \sum_{j=1}^{2n} \xi_j([X_j, X]). \tag{4.1}$$

[5]Si B est connexe, cette condition $Tr_{\mathfrak{g}/\mathfrak{b}}\, ad(X) = 0$ pour $X \in \mathfrak{b}$ équivaut à l'existence d'un volume invariant.

En effet, d'après (1.5),

$$(\pi^*\partial\pi.X)\,\Omega = \pi^*.((\partial\pi.X)\,\omega) = \pi^*.(\theta(\pi.X).\omega) = \theta(X)\,\pi^*.\omega = \theta(X).\Omega.$$

D'autre part,

$$\theta(X)\,.\,\Omega = \sum_{j=1}^{2n} \xi_1 \wedge \ldots (\theta(X)\,.\,\xi_j) \wedge \ldots \xi_{2n}$$

$$= \sum_{j=1}^{2n} \sum_{k=1}^{m} (\theta(X)\,.\,\xi_j)(X_k)\xi_1 \wedge \ldots \xi_{j-1} \wedge \xi_k \wedge \xi_{j+1} \wedge \ldots \xi_{2n}.$$

Puisque les fonctions $\xi_j(X_k)$ sont des constantes, la formule (1.1) montre que

$$(\theta(X).\xi_j)(X_k) - \xi_j([X, X_k]) = \theta(X).(\xi_j(X_k)) = 0.$$

Or, pour $2n < k \leqslant m$, $X_k \in \mathfrak{b}$, donc d'après (1.2), $\pi.[X, X_k] = 0$, c'est à dire que $[X, X_k]$ est dans le module $F\mathfrak{b}$ engendré par $\mathfrak{b}$. Il en résulte que $\xi_j([X, X_k]) = 0$ pour tout $i > 2n$ et par suite,

$$\theta(X)\,.\,\Omega = \Big(\sum_{j=1}^{2n} \xi_j([X_j, X])\Big)\Omega.$$

La formule (4.1) en résulte parceque $\Omega \neq 0$ en tout point de G.

Supposons maintenant donnée sur G/B une structure complexe invariante par G. Cette structure sera définie par un endomorphisme J de l'espace $\mathfrak{g}$ vérifiant les propriétés (2.2), (2.3), (2.4) et (2.6). La restriction aux champs de vecteurs réels de la forme hermitienne canonique h de G/B a pour image inverse sur G une forme bilinéaire symétrique invariante à gauche $\eta = \pi^*.h$. De même, la forme antisymétrique a définie par $a(U, V) = h(U, I.V)$ aura pour image inverse une forme antisymétrique $\alpha = \pi.^*a$ invariante à gauche sur G. Il résulte de (1.4) et (2.5) que

$$\eta(X, Y) = \alpha(J.X, Y), \tag{4.2}$$

quels que soient les champs de vecteurs X, Y sur G. Nous allons procéder au calcul de α.

Soit X un champ de vecteurs invariant *à droite* sur G. A sa projection $\pi.X$ est associée une transformation infinitésimale du groupe G opérant sur G/B Comme ces opérations de G conservent le volume ω et la structure complexe de G/B, $\pi.X$ est un champ de vecteurs conforme et $\partial\pi.X = 0$. Si donc X et Y sont deux champs de vecteurs invariants à droite sur G, on a d'après (3. 3), (1. 2) et (2. 5):

$$2a(\pi.X, \pi.Y) = \partial.(I.[\pi.X, \pi.Y]) = \partial.(I\pi.[X, Y]) = \partial.(\pi J.[X, Y]).$$

D'après (1. 4) et (4. 1), on a donc

$$2\alpha(X, Y) = \sum_{j=1}^{2n} \xi_j([X_j, J\,.\,[X, Y]]). \tag{4.3}$$

De cette formule qui donne $\alpha(X, Y)$ lorsque X et Y sont des champs invariants à droite, il nous reste à déduire la valeur de $\alpha(X, Y)$ lorsque X et Y

sont des champs invariants à *gauche* sur G. Soient X' et Y' les champs invariants à droite qui coïncident respectivement au point neutre e de G avec les champs invariants à gauche X et Y. Posons

$$[X', Y'] = \sum_{k=1}^{m} f_k X_k = T$$

où les f_k sont des fonctions différentiables sur G. Comme T est invariant à droite, on a[3] pour tout indice j:

$$0 = [X_j, T] = \sum_{k=1}^{m} (\theta(X_j) \cdot f_k) X_k + \sum_{k=1}^{m} f_k [X_j, X_k],$$

et par suite,

$$[X_j, J \cdot T] = \sum_{k=1}^{m} (\theta(X_j) \cdot f_k)(J \cdot X_k) + \sum_{k=1}^{m} f_k [X_j, J \cdot X_k].$$

Or le champ invariant à gauche

$$\sum_{k=1}^{m} f_k(e) X_k$$

qui coïncide en e avec $[X', Y']$ est égal à $-[X, Y]$. Par conséquent, au point e, $[X_j, J.T]$ coïncide avec le champ invariant à gauche $J.[X_j, [X, Y]] - [X_j, J.[X, Y]]$. La formule (4.3) donne donc comme valeur de la fonction $\alpha(X, Y)$ au point e:

$$2\alpha(X, Y) = \sum_{j=1}^{2n} \xi_j([J \cdot [X, Y], X_j] - J \cdot [[X, Y], X_j]). \tag{4.4}$$

Comme α est invariante à gauche, cette égalité est vérifiée en tout point de G. Pour interpréter cette formule, on désignera par ψ la forme de degré 1 invariante à gauche sur G qui est définie par

$$\psi(X) = \mathrm{Tr}_{\mathfrak{g}/\mathfrak{h}}(\mathrm{ad}(J.X) - J\,\mathrm{ad}(X)) \tag{4.5}$$

pour tout $X \in \mathfrak{g}$. Cette définition est justifiée car, d'après (2.4), si $Y \in \mathfrak{b}$, $J\,\mathrm{ad}(Y) - \mathrm{ad}(Y)\,J$ applique $\mathfrak{g}$ dans $\mathfrak{b}$ et par suite, l'endomorphisme $\mathrm{ad}(J.X) - J\,\mathrm{ad}(X)$ applique $\mathfrak{b}$ dans $\mathfrak{b}$ quel que soit $X \in \mathfrak{g}$. La formule (4.5) s'écrit alors,

$$2\alpha(X, Y) = \psi([X, Y]) = (\delta.\psi)(X, Y) \tag{4.6}$$

quels que soient $X, Y \in \mathfrak{g}$. D'où le

THÉORÈME 1. *Si h est la forme hermitienne canonique de G/B, l'image inverse sur G de la forme antisymétrique $h(U, I.V)$ est la différentielle extérieure de la forme invariante a gauche ψ que définit* **(4. 5)**.

Cette forme ψ est déterminée sans ambiguité par la structure complexe de G/B. En effet, si J' est un autre endomorphisme de $\mathfrak{g}$ qui définit la même structure complexe sur G/B, on a $J'.X = J.X$ mod $\mathfrak{b}$ pour tout $X \in \mathfrak{g}$ et par suite,

$$\mathrm{Tr}_{\mathfrak{g}/\mathfrak{b}}(\mathrm{ad}(J'.X) - J'\mathrm{ad}(X)) - \mathrm{Tr}_{\mathfrak{g}/\mathfrak{b}}(\mathrm{ad}(J.X) - J\,\mathrm{ad}(X))$$
$$= \mathrm{Tr}_{\mathfrak{g}/\mathfrak{b}}(\mathrm{ad}(J'.X - J.X) - (J' - J)\,\mathrm{ad}(X)) = 0$$

car $\mathrm{Tr}_{\mathfrak{g}/\mathfrak{b}}\,\mathrm{ad}(Y) = 0$ pour tout $Y \in \mathfrak{b}$.

(4.7) *La forme ψ est invariante par les translations à droite par des éléments de B.* Soit en effet $r(s)$ l'automorphisme $X \to Xs$ de l'algèbre de Lie $\mathfrak{g}$ que définit la translation à droite par un élément $s \in G$. La condition (2.4) signifie que, si $t \in B$, alors $Jr(t) - r(t)J$ applique $\mathfrak{g}$ dans $\mathfrak{b}$; par suite

$$\mathrm{Tr}_{\mathfrak{g}/\mathfrak{b}}(\mathrm{ad}(Jr(t).X) - \mathrm{ad}(r(t)J.X)) = 0$$

pour tout $X \in \mathfrak{g}$. Il en résulte par un calcul simple

$$\psi(r(t).X) = \mathrm{Tr}_{\mathfrak{g}/\mathfrak{b}}r(t)(\mathrm{ad}(J.X) - J\,\mathrm{ad}(X))\,r(t)^{-1} = \psi(X),$$

c'est à dire l'invariance de ψ par la translation à droite par t.

5. Les espaces homogenes complexes dont la forme hermitienne canonique est non dégénérée. Soit B' le sous-groupe fermé des $t \in G$ tels que ψ soit invariante par la translation à droite par t. Ce sous-groupe qui contient B (4.7) est le sous-groupe de stabilité de ψ dans la représentation duale de la représentation adjointe $s \to r(s)$. Les $X \in \mathfrak{g}$ qui, en e sont tangents à B' constituent une sous-algèbre $\mathfrak{b}'$ de $\mathfrak{g}$ contenant $\mathfrak{b}$. Pour que $X \in \mathfrak{b}'$, il faut et il suffit que $\theta(X).\ \psi = 0$, autrement dit que $2\alpha(X, Y) = \psi([X, Y]) = 0$ pour tout $Y \in \mathfrak{g}$. Ceci signifie que les $X \in \mathfrak{b}'$ sont les éléments de $\mathfrak{g}$ dont l'image par π' (cf. §2) est un élément de $T(e')$ orthogonal à $T(e')$ pour la forme hermitienne canonique de G/B. On a donc le critère:

(5.1) *Pour que la forme hermitienne canonique de G/B soit non dégénérée, il faut et il suffit que $\mathfrak{b}' = \mathfrak{b}$, c'est à dire que B soit un sous-groupe ouvert du groupe de stabilité de ψ dans la représentation duale de la représentation adjointe de G.*

Si la forme canonique est non dégénérée, l'espace homogène G/B est donc un revêtement de la trajectoire G/B' de ψ dans cette représentation duale.[6]

Si l'on suppose G semi-simple, la forme de Killing définit un isomorphisme entre la représentation adjointe et la représentation duale. De (5.1) résulte alors que

(5.2) *Si G est semi-simple et si la forme hermitienne canonique de G/B est non dégénérée, alors B est un sous-groupe ouvert du centralisateur d'un sous-groupe à un paramètre connexe de G.*

Si la forme ψ est nulle sur $\mathfrak{b}$, alors, ψ étant invariante à droite par B, il existe une forme ϕ de degré 1 sur G/B telle que[7] $\pi^*.\phi = \psi$. Cette forme est invariante

[6] Si B est discret, il doit donc exister une trajectoire *ouverte* dans la représentation duale de la représentation adjointe de G.

[7] La forme ψ est *basique* au sens de (7).

par G et, comme

$$2\pi^*.a = 2\alpha = \delta\psi = \delta\pi^*.\phi = \pi^*\delta.\phi,$$

on a $\delta.\phi = 2a$. Si de plus la forme canonique de G/B est non dégénérée, le volume invariant ω sur G/B sera à un facteur constant près la puissance extérieure n-ième de a donc la dérivée extérieure d'une forme invariante $\phi \wedge a \wedge a \ldots a$. On voit donc que

(5.3) *Si le volume invariant sur G/B n'est pas la différentielle extérieure d'une forme invariante de degré $2n-1$ sur G/B, et si la forme hermitienne canonique de G/B n'est pas dégénérée, alors ψ n'est pas nulle sur $\mathfrak{b}$ et en particulier, $\mathfrak{b} \neq [\mathfrak{b}, \mathfrak{b}]$.*

On sait **(11**, Théorème 12.1**)** que si G est unimodulaire et si la représentation adjointe de B dans $\mathfrak{g}$ est complètement réductible, alors le volume invariant de G/B n'est pas la différentielle d'une forme invariante. De (5.3) résulte donc que

(5.4) *Si G est unimodulaire, si B est compact et si la forme hermitienne canonique de G/B est non dégénérée, alors le centre de B est de dimension*[8] > 0.

Supposons maintenant que G opère de manière effective sur G/B; le sous-groupe B ne contient alors aucun sous-groupe distingué de G autre que (e). Si la forme hermitienne canonique de G/B est non dégénérée, la composante connexe de l'élément neutre dans le centre de G se réduit à (e) car, d'après (5.1) elle est contenue dans B. Le centre de G est donc un sous-groupe *discret*. D'autre part, la forme canonique étant toujours supposée non dégénérée, si la représentation linéaire de B dans $\mathfrak{g}/\mathfrak{b}$ qui est définie par la représentation adjointe de G est une représentation simple, alors G est un groupe semi-simple. En effet, si $\mathfrak{m}$ est un idéal abélien de $\mathfrak{g}$, son image dans le quotient $\mathfrak{g}/\mathfrak{b}$ est soit (0) soit $\mathfrak{g}/\mathfrak{b}$. Dans le premier cas $\mathfrak{m} \subset \mathfrak{b}$ et donc $\mathfrak{m} = (0)$ puisque G est effectif. Dans le second, la formule $2\alpha(X, Y) = \psi([X, Y])$ montre que $\mathfrak{m} \cap \mathfrak{b}' = \mathfrak{b}$, donc que $\mathfrak{g} = (0)$.

6. Le cas d'un groupe transitif semi-simple. Dans ce paragraphe, on supposera que G est un groupe semi-simple et que le sous-groupe B est un sous-groupe ouvert du centralisateur d'un sous-groupe à un paramètre L de G. On supposera en outre que B *est compact*. On se propose de montrer que G/B possède des structures complexes invariantes et d'étudier les formes hermitiennes correspondantes.

Soit M un sous-groupe compact maximal de G contenant B, donc L. Puisque M est connexe **(10)**, le centralisateur de L dans M est un sous-groupe connexe **(9)** de M et il en résulte que B est *connexe*. Soit Z le centre de G et $G_a = G/Z$ le groupe adjoint de $\mathfrak{g}$. L'image canonique de B dans G/Z est un sous-groupe compact B_a de G_a. Soit K_a un sous-groupe compact maximal de G_a contenant

[8] Ce résultat est à rapprocher de certains résultats de Lichnerowicz **(12)**.

B_a et soit K le sous-groupe de G image inverse de K_a dans G. La composante connexe de l'élément neutre dans le centre de K_a est un sous-groupe fermé de B car elle est dans la composante connexe de l'élément neutre du centralisateur de $L \subset K$. Par suite, cette composante connexe du centre de K est compacte. Puisque K est connexe **(10)** et localement isomorphe au groupe compact K_a, il en résulte que K est compact. Ainsi, *le centre Z de G est un sous-groupe fini* de G et les sous-groupes compacts maximaux de G et de G_a ont même dimension.

Soit $\mathfrak{k}$ la sous-algèbre des $X \in \mathfrak{g}$ qui, au point neutre, sont tangents au sous-groupe compact maximal K de G. Puisque $\mathfrak{k}$ est également la sous-algèbre qui correspond au sous-groupe compact maximal K_a de G_a, il existe un automorphisme involutif σ de l'algèbre de Lie $\mathfrak{g}$ tel que $\mathfrak{k}$ soit la sous-algèbre des éléments invariante par σ. De plus, les éléments de l'algèbre de Lie $\mathfrak{g}^C$ complexifiée de $\mathfrak{g}$ qui sont de la forme $X + i\,Y$ avec $\sigma.X = X$ et $\sigma.Y = -Y$ constituent une sous-algèbre de Lie $\mathfrak{g}'$ (sur les nombres réels) qui est une forme compacte **(4)** de $\mathfrak{g}$. Soit $\mathfrak{h}$ une sous-algèbre abélienne maximale de $\mathfrak{g}$ contenant la sous-algèbre de dimension 1 qui correspond à L. On a $\mathfrak{h} \subset \mathfrak{k}$ et $\mathfrak{h}$ est également une sous-algèbre abélienne maximale de $\mathfrak{g}'$. Par suite, la sous-algèbre complexe $\mathfrak{h}^C$ de $\mathfrak{g}^C$ engendrée par $\mathfrak{h}$ est une sous-algèbre de Cartan de $\mathfrak{g}^C$. Soient X_γ des vecteurs propres non nuls correspondant aux racines $\gamma \neq 0$ de $\mathfrak{g}^C$ relatives à cette sous-algèbre de Cartan. Si σ^C désigne le prolongement de σ en automorphisme complexe de $\mathfrak{g}^C$, on a $\sigma^C.H = H$ pour tout $H \in \mathfrak{h}^C$ et par suite $\sigma^C.X_\gamma = \pm X_\gamma$ pour toute racine γ.

Soit G' le groupe semi-simple compact adjoint de $\mathfrak{g}'$. Le sous-groupe compact K_a de G_a s'identifie à un sous-groupe de G' et, dans cette identification, B_a devient le sous-groupe fermé de G' qui correspond à la sous-algèbre $\mathfrak{k} \subset \mathfrak{g}'$. Ce sous-groupe B_a de G' est le centralisateur dans G' du sous-groupe image de L dans $B/Z = B_a \subset G'$. Par conséquent, l'espace homogène G'/B_a possède des structures complexes invariantes en nombre fini **(13)**. Soit J' un endomorphisme de l'espace $\mathfrak{g}'$ définissant une de ces structures complexes et soit J^C son prolongement en endomorphisme complexe de l'espace $\mathfrak{g}^C$. Des conditions (2.2), (2.3), (2.4), et (2.6) vérifiées par J', on déduit que J^C vérifie les conditions

(6.1) $\quad J^C.\mathfrak{b}^C = (0)$

(6.2) $\quad (J^C)^2.X = -X \pmod{\mathfrak{b}^C}$ *pour tout* $X \in \mathfrak{g}^C$,

(6.3) $\quad J^C.[X, Y] = [X, J^C.Y] \pmod{\mathfrak{b}^C}$ *quels que soient* $X \in \mathfrak{b}^C$ *et* $Y \in \mathfrak{g}^C$,

(6.4) $\quad [X, Y] + J^C.[J^C.X, Y] + J^C.[X, J^C.Y] - [J^C.X, J^C.Y] \in \mathfrak{b}^C$ *quels que soient* $X, Y \in \mathfrak{g}^C$

Il résulte de (6.3) que, pour toute racine $\gamma \neq 0$,

$$J^C.X_\gamma = \pm i\,X_\gamma \pmod{\mathfrak{b}^C}.$$

Par suite, $J^C\sigma^C - \sigma^C J^C$ applique $\mathfrak{g}^C$ dans $\mathfrak{b}^C$ et par conséquent, $J^C.\mathfrak{g} \subset \mathfrak{g}$. La restriction de J^C à $\mathfrak{g}$ est un endomorphisme J de l'espace $\mathfrak{g}$ qui vérifie

visiblement les conditions (2.2), (2.3) et (2.6). Il vérifie aussi la condition (2.4) car, *B étant connexe*, cette condition est équivalente à la condition $J.[X, Y] = [X, J.Y]$ (mod $\mathfrak{b}$) quels que soient $X \in \mathfrak{b}$, $Y \in \mathfrak{g}$, laquelle résulte de (6.3). Ainsi, *à toute structure complexe invariante sur G'/B_a est associée une structure complexe invariante sur G/B.* Un raisonnement analogue montre que. réciproquement, *toute structure complexe invariante sur G/B est ainsi associée à une structure complexe invariante sur G'/B_a*

Partant d'une structure complexe invariante sur G'/B_a définie par J' et munissant G/B de la structure complexe invariante associée, nous allons comparer les formes hermitiennes canoniques de G'/B_a et de G/B. D'après (6.3), quels que soit $X \in \mathfrak{g}^C$, l'endomorphisme $\mathrm{ad}(J^C.X) - J^C\mathrm{ad}(X)$ laisse stable $\mathfrak{b}^C$. La trace de l'endomorphisme complexe de $\mathfrak{g}^C/\mathfrak{b}^C$ déduit de $\mathrm{ad}(J^C.X) - J^C\mathrm{ad}(X)$ par passage au quotient est une fonction linéaire complexe de $X \in \mathfrak{g}^C$ que l'on note $\psi^C(X)$. Les formes ψ et ψ' pour G/B et G'/B_a sont respectivement égales aux restrictions de ψ^C à $\mathfrak{g}$ et à $\mathfrak{g}'$. Compte tenu de (4.2) et (4.6), on voit donc que les formes η et η', images inverses des formes hermitiennes canoniques de G/B et G'/B, sont égales respectivement aux restrictions à $\mathfrak{g}$ et $\mathfrak{g}'$ de la forme bilinéaire complexe

$$\eta^C(X, Y) = \tfrac{1}{2}\psi^C([J^C.Y, Y]).$$

Comme σ^C est un automorphisme de $\mathfrak{g}$ et que $\sigma^C J^C - J^C\sigma^C$ applique $\mathfrak{g}^C$ dans $\mathfrak{b}^C$, on voit que $\psi^C(\sigma^C.X) = \psi^C(X)$ pour tout $X \in \mathfrak{g}^C$. La forme η^C est donc invariante par σ^C; en particulier, si $\sigma^C.X = X$ et $\sigma^C.Y = -Y$, on a $\eta^C(X, Y) = 0$. On en déduit que

(6.5) *L'image inverse η de la forme canonique de G/B se déduit de l'image inverse η' de la forme canonique de G'/B_a par les relations:*

$$\begin{aligned}
&\eta(X, Y) = \eta'(X, Y) && \text{si } \sigma.X = X \text{ et } \sigma.Y = Y,\\
&\eta(X, Y) = 0 && \text{si } \sigma.X = X \text{ et } \sigma.Y = -Y,\\
&\eta(X, Y) = -\eta'(iX, iY) && \text{si } \sigma.X = -X \text{ et } \sigma.Y = -Y.
\end{aligned}$$

On est ainsi ramené à l'étude du cas où G est *compact*, $\mathfrak{g}'$ étant donc l'algèbre de Lie $\mathfrak{g}$ elle-même. Puisque $\mathfrak{g}$ est une forme compacte de $\mathfrak{g}^C$, toute racine α relative à $\mathfrak{b}^C$ vérifie $\bar{\alpha} = -\alpha$. Pour toute racine $\alpha \neq 0$, soit H_α' l'élément de $\mathfrak{b}^C$ tel que $\alpha(H) = \mathrm{Tr}\ \mathrm{ad}\ (H_\alpha')\ \mathrm{ad}(H)$ pour tout $H \in \mathfrak{b}^C$. On sait **(14)** que les X_α peuvent être choisis de telle sorte que, pour toute racine $\alpha \neq 0$,

$$(6,6) \qquad \bar{X}_\alpha = -X_{-\alpha}, \quad [X_\alpha, X_{-\alpha}] = H_\alpha \text{ où } H_\alpha = 2\alpha(H_\alpha')^{-1}H_\alpha'$$

Le sous-groupe B étant compact, on peut supposer que l'endomorphisme J de $\mathfrak{g}$ qui définit la structure complexe invariante sur G/B commute avec $\mathrm{ad}(X)$ pour tout $X \in \mathfrak{b}$. Les X_α sont alors des vecteurs propres appartenant aux racines 0, i ou $-i$ de J^C. Soit $\mathscr{D}$ l'ensemble des racines $\alpha \neq 0$ telles que $J^C.X = iX$; les X_α et $X_{-\alpha}$ pour $\alpha \in \mathscr{D}$ constituent une base d'un supplémentaire complexe de $\mathfrak{b}^C$ dans $\mathfrak{g}^C$. Pour toute racine $\gamma \neq 0$, on a $X_\gamma \in [\mathfrak{b}^C, \mathfrak{g}^C]$

et par suite $\psi^C(X_\gamma) = 0$. Pour tout $H \in \mathfrak{h}^C$, on a $J^C.H = 0$ et $J^C \operatorname{ad}(H)$. $X_\alpha = i\,\alpha(H)\,X_\alpha$ lorsque $\alpha \in \mathscr{D}$. La définition de ψ^C donne donc[9]

$$\psi^C(H) = -2i\sum\alpha(H), \qquad \alpha \in \mathscr{D}. \tag{6.7}$$

Les $U_\alpha = X_\alpha - X_{-\alpha}$ et $V_\alpha = i\,(X_\alpha + X_{-\alpha})$ pour $\alpha \in \mathscr{D}$ constituent une base d'un supplémentaire de $\mathfrak{h}$ dans $\mathfrak{g}$. On a $J.U_\alpha = V_\alpha$ et $J.V_\alpha = -U_\alpha$ pour tout $\alpha \in \mathscr{D}$. Si α et β sont des racines distinctes, $[U_\alpha, V_\beta]$ est combinaison linéaire des X_γ appartenant aux racines $\gamma \neq 0$; il en est de même de $[U_\alpha, U_\beta]$ et $[V_\alpha, V_\beta]$ quelles que soient les racines $\alpha, \beta \neq 0$. Puisque $\psi^C(X_\gamma) = 0$ pour $\gamma \neq 0$, il résulte de (4.6) que les U_α et V_β sont deux à deux orthogonaux pour la forme η. Compte tenu de (6.6), on a

$$\eta(U_\alpha, U_\alpha) = \eta(V_\alpha, V_\alpha) = \tfrac{1}{2}\psi([J.U_\alpha, U_\alpha]) = -\tfrac{1}{2}i\,\psi^C(H_\alpha) \tag{6.8}$$

pour toute racine $\alpha \neq 0$.

LEMME 2. *Pour toute racine $\gamma \in \mathscr{D}$, on a $\sum\alpha(H_\gamma) > 0$ et pour toute racine $\gamma \neq 0$ telle que $X_\gamma \in \mathfrak{b}^C$, on a $\sum\alpha(H_\gamma) = 0$, les sommes portant sur toutes les racines $\alpha \in \mathscr{D}$.*

Soit $\mathscr{G}$ l'ensemble de racines comportant la racine 0, les racines dans $\mathscr{D}$ et les racines $\alpha \neq 0$ telles que $X_\alpha \in \mathfrak{b}^C$. Pour qu'une racine α soit dans $\mathscr{G} - \mathscr{D}$ il faut et il suffit que $\alpha \in \mathscr{G}$ et $-\alpha \in \mathscr{G}$. La condition d'intégrabilité (2.6) vérifiée par J signifie que le sous-espace complexe engendré par les vecteurs propres qui appartiennent à une racine dans $\mathscr{G}$ est une sous-algèbre de Lie de $\mathfrak{g}^C$. Par suite, si $\alpha, \beta \in \mathscr{G}$ et si $\alpha + \beta$ est racine, alors $\alpha + \beta \in \mathscr{G}$. Il en résulte que, *si $\alpha \in \mathscr{G}$, $\beta \in \mathscr{D}$ et si $\alpha + \beta$ est racine, alors $\alpha + \beta \in \mathscr{D}$*. On a en effet $\alpha + \beta \in \mathscr{G}$ et si $\alpha + \beta$ était dans $\mathscr{G} - \mathscr{D}$, on aurait $-(\alpha + \beta) \in \mathscr{G}$, donc $-\beta = \alpha - (\alpha + \beta) \in \mathscr{G}$, d'où $\beta \in \mathscr{G} - \mathscr{D}$ contrairement à l'hypothèse. Soit γ une racine $\neq 0$ dans $\mathscr{G}$ et soit β une racine telle que $\beta + \gamma$ ne soit pas racine. On appellera classe de β (relativement à γ) l'ensemble $\mathscr{D}_\beta$ des racines *dans* $\mathscr{D}$ qui sont de la forme $\beta + k\gamma$ (k entier). Comme l'ensemble de toutes les racines de la forme $\beta + k\gamma$ s'obtient pour $p \leqslant k \leqslant 0$ où p est un entier $\leqslant 0$, il résulte de la remarque précédente que, si $\mathscr{D}_\beta \neq \phi$, il existe un entier s vérifiant $p \leqslant s \leqslant 0$ tel que $\mathscr{D}_\beta$ soit l'ensemble des racines de la forme $\beta + k\gamma$ avec $s \leqslant k \leqslant 0$. Si β et γ sont linéairement indépendantes, on sait **(14)** que $\beta(H_\gamma) = -p$. Puisque $\gamma(H_\gamma) = 2$, on a donc dans ce cas

$$\begin{aligned}\sum\alpha(H_\gamma) &= (\beta + \beta - \gamma + \ldots + \beta + s\gamma)(H_\gamma) \qquad (\alpha \in \mathscr{D}_\beta)\\ &= (1 - s)(s - p) \geqslant 0.\end{aligned}$$

Si β et γ sont linéairement dépendantes, alors $\beta = \gamma$, car $\beta = 0$ et $\beta = -\gamma$ sont exclus du fait que $\beta + \gamma$ n'est pas racine. Dans ce cas, si $\mathscr{D}_\beta \neq \phi$, c'est donc que $\gamma \in \mathscr{D}$ et $\mathscr{D}_\beta = (\gamma)$, donc $\sum\alpha(H_\gamma) = 2 > 0$ pour $\alpha \in \mathscr{D}_\beta$. Comme

[9] La restriction de ψ^C à $\mathfrak{h}$ est bien une forme réelle car, si $H = \overline{H}$, on a $\overline{\alpha(H)} = -\alpha(H)$ pour toute α.

toute racine dans $\mathscr{D}$ appartient à une classe et une seule, on peut calculer $\sum\alpha(H_\gamma)$ pour $\alpha \in \mathscr{D}_\gamma$ en groupant les racines α par classes. Si $\gamma \in \mathscr{D}_\gamma$ les sommes partielles sont toutes >0 et la somme partielle relative à $\mathscr{D}_\gamma$ est >0, d'où la première partie du Lemme. Si $\gamma \in \mathscr{G} - \mathscr{D}_\gamma$ alors $-\gamma \in \mathscr{G}$; pour toute racine β telle que $\mathscr{D}_\beta \neq 0$, γ est linéairement indépendante de β et $p = s$. On a donc $\sum\alpha(H_\gamma) = 0$ pour $\alpha \in \mathscr{D}_\beta$, d'où la seconde partie du Lemme[10].

De ce Lemme et des formules (6.7), (6.8), il résulte que la forme hermitienne canonique de G/B est *définie négative*. Compte tenu des relations (6.5) qui ramènent le cas d'un groupe G semi-simple au cas où G est compact, on obtient les Théorèmes suivants:

THÉORÈME 2. *Soit G un groupe semi-simple et soit L un sous-groupe à un paramètre de G. Si un sous-groupe compact B de G est un sous-groupe ouvert du centralisateur de L, alors B est connexe et l'espace homogène G/B possède des structures complexes invariantes. Pour chacune de ces structures la forme hermitienne canonique de G/B est non dégénérée. Le nombre de ses carrés négatifs est égal à la différence entre la dimension des sous-groupes compacts maximaux de G et la dimension de B.*

THÉORÈME 3. *Si G est un groupe compact et si G/B est un espace homogène complexe dont la forme hermitienne canonique est non dégénérée, alors cette forme est définie négative.*

En effet, la composante connexe de l'élément neutre dans le centre de G opère trivialement dans G/B d'après les résultats du §5. On peut donc supposer G semi-simple, et la remarque (5.2) permet d'appliquer[11] le Théorème 2.

THÉORÈME 4. *Soient G un groupe semi-simple et B un sous-groupe compact de G. S'il existe sur l'espace homogène G/B une structure complexe invariante pour laquelle la forme hermitienne canonique est non dégénérée, alors B est connexe, le centre de G est fini et le nombre des carrés négatifs de la forme canonique est égal à la différence entre la dimension des sous-groupes compacts maximaux de G et la dimension de B.*

C'est encore une conséquence de la remarque (5.2) et du Théorème 2. En particulier, il résulte du Théorème 4 que si la forme hermitienne canonique

[10]Nous n'utiliserons ici que la premiere partie du Lemme. La seconde a l'intéret de montrer que les racines $\gamma \leqslant \mathscr{D}$ peuvent être caractérisées par une inégalité de la forme $\lambda(H_\gamma) \leqslant 0$ où λ est une fonction linéaire sur $\mathfrak{h}^c$. On obtient ainsi une manière de déterminer toutes les structures complexes invariantes sur G/B (3; 13).

[11]D'après un Théorème de Montgomery (*Simply connected homogeneous spaces*, Proc. Amer. Math. Soc., *1* (1950), 467–469), si G/B est compact et a un groupe de Poincaré fini, tout sous-groupe compact maximal dans G opère transitivement sur G/B. Par suite, si un espace homogène complexe compact a un groupe de Poincaré fini et si sa forme hermitienne canonique est non dégénérée, alors cette forme est définie négative. Les Théorèmes 3 et 4 sont en rapports étroits avec les résultats de Bochner et Lichnerowicz sur les champs de vecteurs de Killing et le tenseur de Ricci (2; 12).

est définie positive, alors B est un sous-groupe compact maximal et G/B est un espace riemannien symétrique. Ceci montre que *les domaines bornés homogènes dont le groupe des automorphismes est semi-simple sont des domaines bornés symétriques au sens de E. Cartan* **(5)**; on sait en effet que leur forme hermitienne canonique est définie positive **(1)** et que le groupe de stabilité d'un point est compact **(6)**. Une démonstration plus directe de ce résultat a été donnée par A. Borel qui suppose seulement que le groupe de tous les automorphismes du domaine borné contient un sous-groupe transitif semi-simple **(3)**[12].

[12]On peut conjecturer que la non dégénérescence de la forme hermitienne canonique d'un espace homogène complexe G/B correspond à une propriété intrinsèque du plus grand groupe des automorphismes analytiques de G/B qui conservent le volume (peut être sa semi-simplicité). D'une manière générale, il serait utile de savoir jusqu'où va l'analogie que nos résultats semblent indiquer entre la forme de Killing d'un groupe et la forme hermitienne canonique d'un espace homogène complexe.

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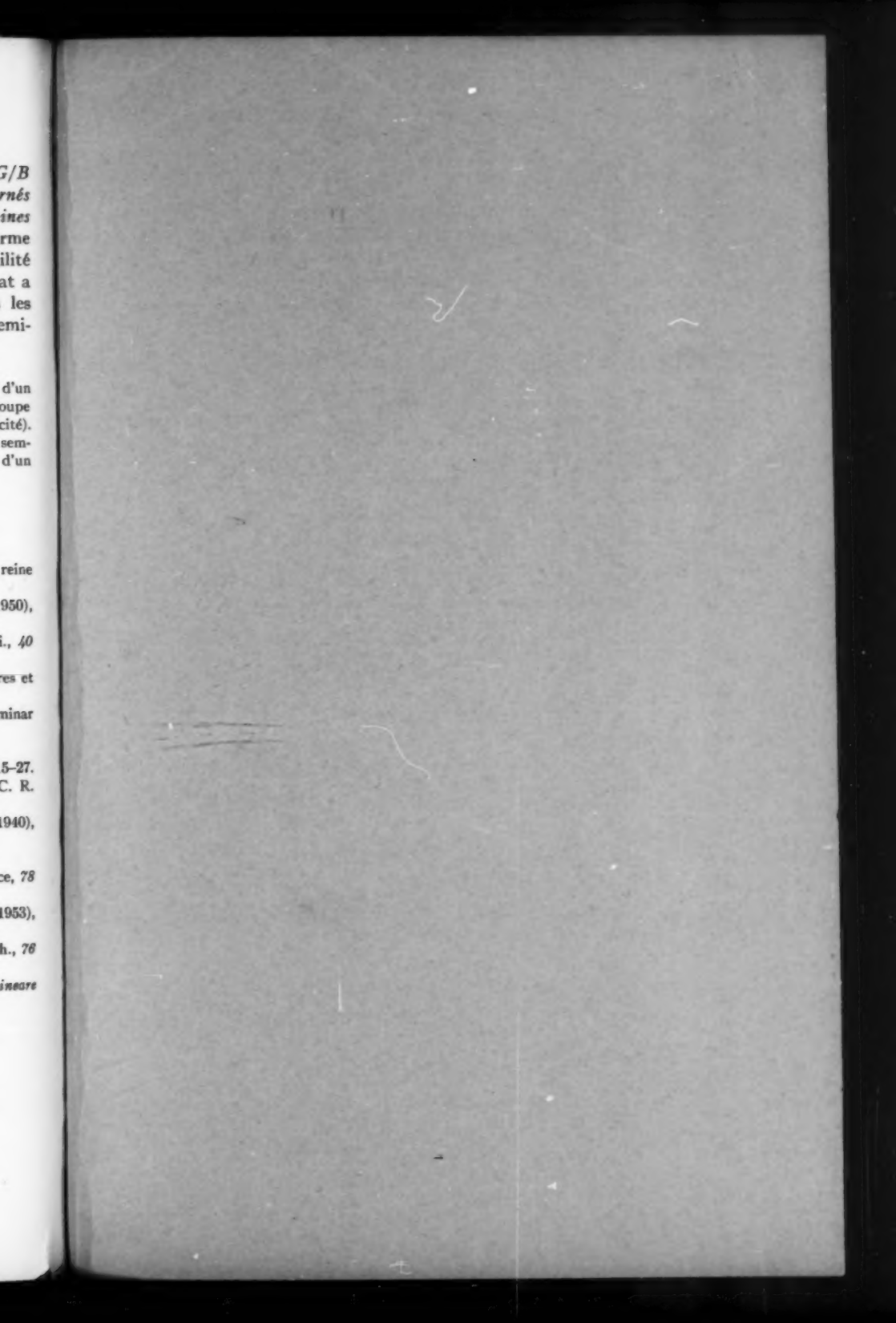

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